

Generalization of Doob Decomposition Theorem and Risk Assessment in Incomplete Markets

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Abstract

In the paper, we introduce the notion of a local regular supermartingale relative to a convex set of equivalent measures and prove for it the necessary and sufficient conditions of optional Doob decomposition in the discrete case. This Theorem is a generalization of the famous Doob decomposition onto the case of supermartingales relative to a convex set of equivalent measures. The description of all local regular supermartingales relative to a convex set of equivalent measures is presented. A notion of complete set of equivalent measures is introduced. We prove that every non negative bounded supermartingale relative to a complete set of equivalent measures is local regular. A new definition of fair price of contingent claim in incomplete market is given and a formula for fair price of Standard option of European type is found.

Keywords: random process, convex set of equivalent measures, optional Doob decomposition, regular supermartingale, martingale, fair price of contingent claim
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1 Introduction.

In the paper, martingales and supermartingales relative to a convex set of equivalent measures are systematically studied. The notion of local regular supermartingale relative to a convex set of equivalent measures is introduced and the necessary and sufficient are found under that a supermartingale is local regular one. Complete description of local regular supermartingales is given. The notion of complete convex set of equivalent measures is introduced and established that every nonnegative supermartingale is local regular relative this set of measures. The notion of local regular supermartingale is used for definition of fair price of contingent claim relative to a convex set of equivalent measures. Sufficient conditions of the existence of fair price of contingent claim relative to a convex set of equivalent measures are presented. All these notions are used in the case as a convex set of equivalent measures is a set of equivalent martingale measures for evolution as risk and non risk assets. Formulas for fair price of standard contract with option of European type in incomplete are found.

The notion of complete convex set of equivalent measures permits to give a new proof of optional decomposition for non negative supermartingale. This proof do not use no-arbitrage arguments and measurable choice [15], [7], [6], [8].

First, optional decomposition for supermartingale was opened by by El Karoui N. and Quenez M. C. [5] for diffusion processes. After that, Kramkov D. O. and Follmer H. [15], [7] proved the optional decomposition for nonnegative bounded supermartingales. Folmer H. and Kabanov Yu. M. [6], [8] proved analogous result for an arbitrary supermartingale. Recently, Bouchard B. and Nutz M. [1] considered a class of discrete models and proved the necessary and sufficient conditions for validity of optional decomposition.

The optional decomposition for supermartingales plays fundamental role for risk assessment in incomplete markets [5], [15], [7], [9], [10], [11]. Considered in the paper problem is generalization of corresponding one that appeared in mathematical

finance about optional decomposition for supermartingale and which is related with construction of superhedge strategy in incomplete financial markets.

Our statement of the problem unlike the above-mentioned one and it is more general: a supermartingale relative to a convex set of equivalent measures is given and it is necessary to find conditions on the supermartingale and the set of measures under that optional decomposition exists.

Generality of our statement of the problem is that we do not require that the considered set of measures was generated by random process that is a local martingale as it is done in the papers [1, 5, 8, 15] and that is important for the proof of the optional decomposition in these papers.

2 Optional decomposition for supermartingales relative to a convex set of equivalent measures.

We assume that on a measurable space $\{\Omega, \mathcal{F}\}$ a filtration $\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}$, $m = \overline{0, \infty}$, and a family of measures M on \mathcal{F} are given. Further, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A random process $\psi = \{\psi_m\}_{m=0}^\infty$ is said to be adapted one relative to the filtration $\{\mathcal{F}_m\}_{m=0}^\infty$ if ψ_m is \mathcal{F}_m measurable random value for all $m = \overline{0, \infty}$.

Definition 2.1 *An adapted random process $f = \{f_m\}_{m=0}^\infty$ is said to be a supermartingale relative to the filtration \mathcal{F}_m , $m = \overline{0, \infty}$, and the family of measures M if $E^P|f_m| < \infty$, $m = \overline{1, \infty}$, $P \in M$, and the inequalities*

$$E^P\{f_m|\mathcal{F}_k\} \leq f_k, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad P \in M, \quad (2.1)$$

are valid.

We consider that the filtration \mathcal{F}_m , $m = \overline{0, \infty}$, is fixed. Further, for a supermartingale f we use as denotation $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ and denotation $\{f_m\}_{m=0}^\infty$.

Below, in a few theorems, we consider a convex set of equivalent measures M satisfying conditions: Radon – Nicodym derivative of any measure $Q_1 \in M$ with respect to any measure $Q_2 \in M$ satisfies inequalities

$$0 < l \leq \frac{dQ_1}{dQ_2} \leq L < \infty, \quad Q_1, Q_2 \in M, \quad (2.2)$$

where real numbers l , L do not depend on Q_1 , $Q_2 \in M$.

Theorem 2.1 *Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a supermartingale concerning a convex set of equivalent measures M satisfying conditions (2.2). If for a certain measure $P_1 \in M$ there exist a natural number $1 \leq m_0 < \infty$, and \mathcal{F}_{m_0-1} measurable nonnegative random value φ_{m_0} , $P_1(\varphi_{m_0} > 0) > 0$, such that the inequality*

$$f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0},$$

is valid, then

$$f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \frac{l}{1+L}\varphi_{m_0}, \quad Q \in M_{\bar{\varepsilon}_0},$$

where

$$M_{\bar{\varepsilon}_0} = \{Q \in M, Q = (1 - \alpha)P_1 + \alpha P_2, 0 \leq \alpha \leq \bar{\varepsilon}_0, P_2 \in M\}, \quad P_1 \in M,$$

$$\bar{\varepsilon}_0 = \frac{L}{1+L}.$$

Proof. Let $B \in \mathcal{F}_{m_0-1}$ and $Q = (1 - \alpha)P_1 + \alpha P_2$, $P_2 \in M$, $0 < \alpha < 1$. Then

$$\begin{aligned}
& \int_B [f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dQ = \\
& \int_B E^Q\{[f_{m_0-1} - f_{m_0}]\mathcal{F}_{m_0-1}\}dQ = \\
& \int_B [f_{m_0-1} - f_{m_0}]dQ = \\
& (1 - \alpha) \int_B [f_{m_0-1} - f_{m_0}]dP_1 + \\
& \alpha \int_B [f_{m_0-1} - f_{m_0}]dP_2 = \\
& (1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_1 + \\
& \alpha \int_B [f_{m_0-1} - E^{P_2}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_2 \geq \\
& (1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_1 = \\
& (1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]\frac{dP_1}{dQ}dQ \geq \\
& (1 - \alpha)l \int_B \varphi_{m_0}dQ \geq (1 - \bar{\varepsilon}_0)l \int_B \varphi_{m_0}dQ = \frac{l}{1 + L} \int_B \varphi_{m_0}dQ.
\end{aligned}$$

Arbitrariness of $B \in \mathcal{F}_{m_0-1}$ proves the needed inequality.

Lemma 2.1 *Any supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a family of measures M for which there hold equalities $E^P f_m = f_0$, $m = \overline{1, \infty}$, $P \in M$, is a martingale with respect to this family of measures and the filtration \mathcal{F}_m , $m = \overline{1, \infty}$.*

Proof. The proof of Lemma 2.1 see [16].

Remark 2.1 *If the conditions of Lemma 2.1 are valid, then there hold equalities*

$$E^P\{f_m|\mathcal{F}_k\} = f_k, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad P \in M. \quad (2.3)$$

Let $f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a supermartingale relative to a convex set of equivalent measures M and the filtration \mathcal{F}_m , $m = \overline{0, \infty}$. And let G be a set of adapted non-decreasing processes $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, such that $f + g = \{f_m + g_m\}_{m=0}^\infty$ is a supermartingale concerning the family of measures M and the filtration \mathcal{F}_m , $m = \overline{0, \infty}$.

Introduce a partial ordering \preceq in the set of adapted non-decreasing processes G .

Definition 2.2 *We say that an adapted non-decreasing process $g_1 = \{g_m^1\}_{m=0}^\infty$, $g_0^1 = 0$, $g_1 \in G$, does not exceed an adapted non-decreasing process $g_2 = \{g_m^2\}_{m=0}^\infty$, $g_0^2 = 0$, $g_2 \in G$, if $P(g_m^2 - g_m^1 \geq 0) = 1$, $m = \overline{1, \infty}$. This partial ordering we denote by $g_1 \preceq g_2$.*

For every nonnegative adapted non-decreasing process $g = \{g_m\}_{m=0}^\infty \in G$ there exists limit $\lim_{m \rightarrow \infty} g_m$ which we denote by g_∞ .

Lemma 2.2 *Let \tilde{G} be a maximal chain in G and for a certain $Q \in M$ $\sup_{g \in \tilde{G}} E_1^Q g = \alpha^Q < \infty$. Then there exists a sequence $g^s = \{g_m^s\}_{m=0}^\infty \in \tilde{G}$, $s = 1, 2, \dots$, such that*

$$\sup_{g \in \tilde{G}} E_1^Q g = \sup_{s \geq 1} E_1^Q g^s,$$

where

$$E_1^Q g = \sum_{m=0}^{\infty} \frac{E^Q g_m}{2^m}, \quad g \in G.$$

Proof.

Let $0 < \varepsilon_s < \alpha^Q$, $s = \overline{1, \infty}$, be a sequence of real numbers satisfying conditions $\varepsilon_s > \varepsilon_{s+1}$, $\varepsilon_s \rightarrow 0$, as $s \rightarrow \infty$. Then there exists an element $g^s \in \tilde{G}$ such that $\alpha^Q - \varepsilon_s < E_1^Q g^s \leq \alpha^Q$, $s = \overline{1, \infty}$. The sequence $g^s \in \tilde{G}$, $s = \overline{1, \infty}$, satisfies Lemma 2.2 conditions.

Lemma 2.3 *If a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M is such that*

$$|f_m| \leq \varphi, \quad m = \overline{0, \infty}, \quad E^Q \varphi < T < \infty, \quad Q \in M, \quad (2.4)$$

where a real number T does not depend on $Q \in M$, then every maximal chain $\tilde{G} \subseteq G$ contains a maximal element.

Proof. Let $g = \{g_m\}_{m=0}^\infty$ belong to G , then

$$E^Q(f_m + \varphi + g_m) \leq f_0 + T, \quad m = \overline{1, \infty}, \quad Q \in M.$$

Then inequalities $f_m + \varphi \geq 0$, $m = \overline{1, \infty}$, yield

$$E^Q g_m \leq f_0 + T, \quad m = \overline{1, \infty}, \quad \{g_m\}_{m=0}^\infty \in G.$$

Introduce for a certain $Q \in M$ an expectation for $g = \{g_m\}_{m=0}^\infty \in G$

$$E_1^Q g = \sum_{m=0}^{\infty} \frac{E^Q g_m}{2^m}, \quad g \in G.$$

Let $\tilde{G} \subseteq G$ be a certain maximal chain. Therefore, we have inequality

$$\sup_{g \in \tilde{G}} E_1^Q g = \alpha_0^Q \leq f_0 + T < \infty,$$

where $Q \in M$ and is fixed. Due to Lemma 2.2,

$$\sup_{g \in \tilde{G}} E_1^Q g = \sup_{s \geq 1} E_1^Q g^s.$$

In consequence of the linear ordering of elements of \tilde{G} ,

$$\max_{1 \leq s \leq k} g^s = g^{s_0(k)}, \quad 1 \leq s_0(k) \leq k,$$

where $s_0(k)$ is one of elements of the set $\{1, 2, \dots, k\}$ on which the considered maximum is reached, that is, $1 \leq s_0(k) \leq k$, and, moreover,

$$g^{s_0(k)} \preceq g^{s_0(k+1)}.$$

It is evident that

$$\max_{1 \leq s \leq k} E_1^Q g^s = E_1^Q g^{s_0(k)}.$$

So, we obtain

$$\sup_{s \geq 1} E_1^Q g^s = \lim_{k \rightarrow \infty} \max_{1 \leq s \leq k} E_1^Q g^s = \lim_{k \rightarrow \infty} E_1^Q g^{s_0(k)} = E_1^Q \lim_{k \rightarrow \infty} g^{s_0(k)} = E_1^Q g^0,$$

where $g^0 = \lim_{k \rightarrow \infty} g^{s_0(k)}$, and that there exists, due to monotony of $g^{s_0(k)}$. Thus,

$$\sup_{s \geq 1} E_1^Q g^s = E_1^Q g^0 = \alpha_0^Q.$$

Show that $g^0 = \{g_m^0\}_{m=0}^\infty$ is a maximal element in \tilde{G} . It is evident that g^0 belongs to G . For every element $g = \{g_m\}_{m=0}^\infty \in \tilde{G}$ two cases are possible:

- 1) $\exists k$ such that $g \preceq g^{s_0(k)}$.
- 2) $\forall k \quad g^{s_0(k)} \prec g$.

In the first case $g \preceq g^0$. In the second one from 2) we have $g^0 \preceq g$. At the same time

$$E_1^Q g^{s_0(k)} \leq E_1^Q g. \quad (2.5)$$

By passing to the limit in (2.5), we obtain

$$E_1^Q g^0 \leq E_1^Q g. \quad (2.6)$$

The strict inequality in (2.6) is impossible, since $E_1^Q g^0 = \sup_{g \in \tilde{G}} E_1^Q g$. Therefore,

$$E_1^Q g^0 = E_1^Q g. \quad (2.7)$$

The inequality $g^0 \preceq g$ and the equality (2.7) imply that $g = g^0$.

Let M be a convex set of equivalent probability measures on $\{\Omega, \mathcal{F}\}$. Introduce into M a metric $|Q_1 - Q_2| = \sup_{A \in \mathcal{F}} |Q_1(A) - Q_2(A)|$, $Q_1, Q_2 \in M$.

Lemma 2.4 *Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a supermartingale relative to a compact convex set of equivalent measures M satisfying conditions (2.2). If for every set of measures $\{P_1, P_2, \dots, P_s\}$, $s < \infty$, $P_i \in M$, $i = \overline{1, s}$, there exist a natural number $1 \leq m_0 < \infty$, and depending on this set of measures \mathcal{F}_{m_0-1} measurable nonnegative random variable $\Delta_{m_0}^s$, $P_1(\Delta_{m_0}^s > 0) > 0$, satisfying conditions*

$$f_{m_0-1} - E^{P_i}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \Delta_{m_0}^s, \quad i = \overline{1, s}, \quad (2.8)$$

then the set G of adapted non-decreasing processes $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to the set of measures M contains nonzero element.

Proof. For any point $P_0 \in M$ let us define a set of measures

$$M^{P_0, \bar{\varepsilon}_0} = \{Q \in M, Q = (1 - \alpha)P_0 + \alpha P, P \in M, 0 \leq \alpha \leq \bar{\varepsilon}_0\}, \quad (2.9)$$

$$\bar{\varepsilon}_0 = \frac{L}{1 + L}.$$

Prove that the set of measures $M^{P_0, \bar{\varepsilon}_0}$ contains some ball of a positive radius, that is, there exists a real number $\rho_0 > 0$ such that $M^{P_0, \bar{\varepsilon}_0} \supseteq C(P_0, \rho_0)$, where $C(P_0, \rho_0) = \{P \in M, |P_0 - P| < \rho_0\}$.

Let $C(P_0, \tilde{\rho}) = \{P \in M, |P_0 - P| < \tilde{\rho}\}$ be an open ball in M with the center at the point P_0 of a radius $0 < \tilde{\rho} < 1$. Consider a map of the set M into itself given by the law: $f(P) = (1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P$, $P \in M$.

The mapping $f(P)$ maps an open ball $C(P'_2, \delta) = \{P \in M, |P'_2 - P| < \delta\}$ with the center at the point P'_2 of a radius $\delta > 0$ into an open ball with the center at the point $(1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P'_2$ of the radius $\bar{\varepsilon}_0 \delta$, since $|(1 - \bar{\varepsilon}_0)P_0 + \bar{\varepsilon}_0 P'_2 - (1 - \bar{\varepsilon}_0)P_0 - \bar{\varepsilon}_0 P| = \bar{\varepsilon}_0 |P'_2 - P| < \bar{\varepsilon}_0 \delta$. Therefore, an image of an open set $M_0 \subseteq M$ is an open set $f(M_0) \subseteq M$, thus $f(P)$ is an open mapping. Since $f(P_0) = P_0$, then the image of the ball $C(P_0, \tilde{\rho}) = \{P \in M, |P_0 - P| < \tilde{\rho}\}$ is a ball $C(P_0, \bar{\varepsilon}_0 \tilde{\rho}) = \{P \in M, |P_0 - P| < \bar{\varepsilon}_0 \tilde{\rho}\}$ and it is contained in $f(M)$. Thus, inclusions $M^{P_0, \bar{\varepsilon}_0} \supseteq f(M) \supseteq C(P_0, \bar{\varepsilon}_0 \tilde{\rho})$ are valid. Let us put $\bar{\varepsilon}_0 \tilde{\rho} = \rho_0$. Then we have $M^{P_0, \bar{\varepsilon}_0} \supseteq C(P_0, \rho_0)$, where $C(P_0, \rho_0) = \{P \in M, |P_0 - P| < \rho_0\}$. Consider an open covering $\bigcup_{P_0 \in M} C(P_0, \rho_0)$ of the compact set M .

Due to compactness of M , there exists a finite subcovering

$$M = \bigcup_{i=1}^v C(P_0^i, \rho_0) \quad (2.10)$$

with the center at the points $P_0^i \in M$, $i = \overline{1, v}$, and a covering by sets $M^{P_0^i, \bar{\varepsilon}_0} \supseteq C(P_0^i, \rho_0)$, $i = \overline{1, v}$,

$$M = \bigcup_{i=1}^v M^{P_0^i, \bar{\varepsilon}_0}. \quad (2.11)$$

Consider the set of measures $P_0^i \in M$, $i = \overline{1, v}$. From Lemma 2.4 conditions, there exist a natural number $1 \leq m_0 < \infty$, and depending on the set of measures $P_0^i \in M$, $i = \overline{1, v}$, \mathcal{F}_{m_0-1} measurable nonnegative random variable $\Delta_{m_0}^v$, $P_0^1(\Delta_{m_0}^v > 0) > 0$, such that

$$f_{m_0-1} - E^{P_0^i}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \Delta_{m_0}^v, \quad i = \overline{1, v}. \quad (2.12)$$

Due to Theorem 2.1, we have

$$f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \frac{l}{1+L}\Delta_{m_0}^v = \varphi_{m_0}^v, \quad Q \in M. \quad (2.13)$$

The last inequality imply

$$E^Q\{f_{m_0-1}|\mathcal{F}_s\} - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0. \quad (2.14)$$

But $E^Q\{f_{m_0-1}|\mathcal{F}_s\} \leq f_s$, $s < m_0$. Therefore,

$$f_s - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0. \quad (2.15)$$

Since

$$f_{m_0} - E^Q\{f_m|\mathcal{F}_{m_0}\} \geq 0, \quad Q \in M, \quad m \geq m_0, \quad (2.16)$$

we have

$$E^Q\{f_{m_0}|\mathcal{F}_s\} - E^Q\{f_m|\mathcal{F}_s\} \geq 0, \quad Q \in M, \quad s < m_0, \quad m \geq m_0. \quad (2.17)$$

Adding (2.17) to (2.15), we obtain

$$f_s - E^Q\{f_m|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0, \quad m \geq m_0, \quad (2.18)$$

or

$$f_s - E^Q\{f_m|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}\chi_{[m_0,\infty)}(m) - \varphi_{m_0}^v\chi_{[m_0,\infty)}(s), \quad (2.19)$$

$$Q \in M, \quad s \leq m_0, \quad m \geq m_0.$$

Introduce an adapted non-decreasing process

$$g^{m_0} = \{g_m^{m_0}\}_{m=0}^\infty, \quad g_m^{m_0} = \varphi_{m_0}^v\chi_{[m_0,\infty)}(m),$$

where $\chi_{[m_0,\infty)}(m)$ is an indicator function of the set $[m_0, \infty)$. Then (2.19) implies that

$$E^Q\{f_m + g_m^{m_0}|\mathcal{F}_k\} \leq f_k + g_k^{m_0}, \quad 0 \leq k \leq m, \quad Q \in M.$$

In the Theorem 2.2 a convex set of equivalent measures

$$M = \{Q, \quad Q = \sum_{i=1}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = \overline{1, n}, \quad \sum_{i=1}^n \alpha_i = 1\} \quad (2.20)$$

satisfies conditions

$$0 < l \leq \frac{dP_i}{dP_j} \leq L < \infty, \quad i, j = \overline{1, n}, \quad (2.21)$$

where l, L are real numbers.

Denote by G the set of all adapted non-decreasing processes $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, such that $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from M .

Theorem 2.2 *Let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to the set of measures (2.20) satisfy the conditions (2.4), and let there exist a natural number $1 \leq m_0 < \infty$, and \mathcal{F}_{m_0-1} measurable nonnegative random value $\varphi_{m_0}^n$, $P_1(\varphi_{m_0}^n > 0) > 0$, such that*

$$f_{m_0-1} - E^{P_i}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^n, \quad i = \overline{1, n}. \quad (2.22)$$

If for the maximal element $g^0 = \{g_m^0\}_{m=0}^\infty$ in a certain maximal chain $\tilde{G} \subseteq G$ the equalities

$$E^{P_i}(f_\infty + g_\infty^0) = f_0, \quad P_i \in M, \quad i = \overline{1, n}, \quad (2.23)$$

are valid, where $f_\infty = \lim_{m \rightarrow \infty} f_m$, $g_\infty^0 = \lim_{m \rightarrow \infty} g_m^0$, then there hold equalities

$$E^P\{f_m + g_m^0|\mathcal{F}_k\} = f_k + g_k^0, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad P \in M. \quad (2.24)$$

Proof. The set M is compact one in the introduced metric topology. From the inequalities (2.22) and the formula

$$E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} = \frac{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_{m_0-1}\} E^{P_i}\{f_{m_0}|\mathcal{F}_{m_0-1}\}}{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_{m_0-1}\}}, \quad Q \in M, \quad (2.25)$$

where $\varphi_i = \frac{dP_i}{dP_1}$, we obtain

$$f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^n, \quad Q \in M. \quad (2.26)$$

The inequalities (2.21) lead to inequalities

$$\frac{1}{nL} \leq \frac{dQ}{dP} \leq nL, \quad P, Q \in M. \quad (2.27)$$

Inequalities (2.26) and (2.27) imply that conditions of Lemma 2.4 are satisfied for any set of measures $Q_1, \dots, Q_s \in M$. Hence, it follows that the set G contains nonzero element. Let $\tilde{G} \subseteq G$ be a maximal chain in G satisfying condition of Theorem 2.2. Denote by $g^0 = \{g_m^0\}_{m=0}^\infty$, $g_0^0 = 0$, a maximal element in $\tilde{G} \subseteq G$. Theorem 2.2 and Lemma 2.3 yield that as $\{f_m\}_{m=0}^\infty$ and $\{g_m^0\}_{m=0}^\infty$ are uniformly integrable relative to each measure from M . There exist therefore limits

$$\lim_{m \rightarrow \infty} f_m = f_\infty, \quad \lim_{m \rightarrow \infty} g_m^0 = g_\infty^0$$

with probability 1. Due to Theorem 2.2 condition, in this maximal chain

$$E^{P_i}(f_\infty + g_\infty^0) = f_0, \quad P_i \in M, \quad i = \overline{1, n}.$$

Since $\{f_m + g_m^0\}_{m=0}^\infty$ is a supermartingale concerning all measures from M , we have

$$E^{P_i}(f_m + g_m^0) \leq E^{P_i}(f_k + g_k^0) \leq f_0, \quad k < m, \quad m = \overline{1, \infty}, \quad P_i \in M. \quad (2.28)$$

By passing to the limit in (2.28), as $m \rightarrow \infty$, we obtain

$$f_0 = E^{P_i}(f_\infty + g_\infty^0) \leq E^{P_i}(f_k + g_k^0) \leq f_0, \quad k = \overline{1, \infty}, \quad P_i \in M. \quad (2.29)$$

So, $E^{P_i}(f_k + g_k^0) = f_0$, $k = \overline{1, \infty}$, $P_i \in M$, $i = \overline{1, n}$. Taking into account Remark 2.1 we have

$$E^{P_i}\{f_m + g_m^0 | \mathcal{F}_k\} = f_k + g_k^0, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad (2.30)$$

$$P_i \in M, \quad i = \overline{1, n}.$$

Hence,

$$\begin{aligned} E^P\{f_m + g_m^0 | \mathcal{F}_k\} = \\ \frac{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_k\} E^{P_i}\{f_m + g_m^0 | \mathcal{F}_k\}}{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_k\}} = f_k + g_k^0, \quad 0 \leq k \leq m, \end{aligned} \quad (2.31)$$

$$P \in M,$$

where $\varphi_i = \frac{dP_i}{dP_1}$, $i = \overline{1, n}$. Theorem 2.2 is proved.

Let M be a convex set of equivalent measures. Below, G_s is a set of adapted non-decreasing processes $\{g_m\}_{m=0}^\infty$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from

$$\hat{M}_s = \{Q, Q = \sum_{i=1}^s \gamma_i \hat{P}_i, \gamma_i \geq 0, i = \overline{1, s}, \sum_{i=1}^s \gamma_i = 1\}, \quad (2.32)$$

where $\hat{P}_1, \dots, \hat{P}_s \in M$ and satisfy conditions

$$0 < l \leq \frac{d\hat{P}_i}{d\hat{P}_j} \leq L < \infty, \quad i, j = \overline{1, s}, \quad (2.33)$$

l, L are real numbers depending on the set of measures $\hat{P}_1, \dots, \hat{P}_s \in M$.

Definition 2.3 Let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M satisfy conditions (2.4). We call it regular one if for every set of measures (2.32) satisfying conditions (2.33) there exist a natural number $1 \leq m_0 < \infty$, and \mathcal{F}_{m_0-1} measurable nonnegative random value $\varphi_{m_0}^s$, $\hat{P}_1(\varphi_{m_0}^s > 0) > 0$, such that the inequalities

$$f_{m_0-1} - E^{\hat{P}_i}\{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^s, \quad i = \overline{1, s},$$

hold and for the maximal element $g^s = \{g_m^s\}_{m=0}^\infty$ in a certain maximal chain $\tilde{G}_s \subseteq G_s$ the equalities

$$E^{\hat{P}_i}\{f_m + g_m^s | \mathcal{F}_k\} = f_k + g_k^s, \quad 0 \leq k \leq m, \quad i = \overline{1, s}, \quad m = \overline{1, \infty}, \quad (2.34)$$

are valid. Moreover, there exists an adapted nonnegative process $\bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^\infty$, $\bar{g}_0^0 = 0$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, $P \in M$, not depending on the set of measures $\hat{P}_1, \dots, \hat{P}_s$ such that

$$E^{\hat{P}_i}\{g_m^s - g_{m-1}^s | \mathcal{F}_{m-1}\} = E^{\hat{P}_i}\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad i = \overline{1, s}. \quad (2.35)$$

The next Theorem describes regular supermartingales.

Theorem 2.3 *Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a regular supermartingale relative to a convex set of equivalent measures M . Then for the maximal element $g^0 = \{g_m^0\}_{m=0}^\infty$ in a certain maximal chain $\tilde{G} \subseteq G$ the equalities*

$$E^{P_0}(f_m + g_m^0) = f_0, \quad m = \overline{1, \infty}, \quad P_0 \in M,$$

are valid. There exists a martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to the family of measures M such that

$$f_m = \bar{M}_m - g_m^0, \quad m = \overline{1, \infty}.$$

Moreover, for the martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ the representation

$$\bar{M}_m = E^{P_0}\{f_\infty + g_\infty | \mathcal{F}_m\}, \quad m = \overline{1, \infty}, \quad P_0 \in M,$$

holds, where $f_\infty + g_\infty = \lim_{m \rightarrow \infty} (f_m + g_m)$.

Proof. For any finite set of measures P_1, \dots, P_n , $P_i \in M$, $i = \overline{1, n}$, let us introduce into consideration two sets of measures

$$M_n = \{P, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i \geq 0, i = \overline{1, n}, \sum_{i=1}^n \alpha_i = 1\},$$

$$\tilde{M}_n = \{P, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i > 0, i = \overline{1, n}, \sum_{i=1}^n \alpha_i = 1\}.$$

Let $\hat{P}_1, \dots, \hat{P}_s$ be a certain subset of measures from \tilde{M}_n . For every measure $\hat{P}_i \in \tilde{M}_n$ the representation $\hat{P}_i = \sum_{k=1}^n \alpha_k^i P_k$ is valid, where $\alpha_k^i > 0$, $i = \overline{1, s}$, $k = \overline{1, n}$. The representation for \hat{P}_i , $i = \overline{1, s}$, imply the validity of inequalities

$$0 < l = \min_{i,j} \frac{\min_k \alpha_k^i}{\max_k \alpha_k^j} \leq \frac{d\hat{P}_i}{d\hat{P}_j} \leq \max_{i,j} \frac{\max_k \alpha_k^i}{\min_k \alpha_k^j} = L < \infty, \quad i, j = \overline{1, s}.$$

Denote by G_s a set of adapted non-decreasing processes $\{g_m\}_{m=0}^\infty$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from

$$\hat{M}_s = \{Q, Q = \sum_{i=1}^s \gamma_i \hat{P}_i, \gamma_i \geq 0, i = \overline{1, s}, \sum_{i=1}^s \gamma_i = 1\}.$$

In accordance with the definition of a regular supermartingale, there exist a natural number $1 \leq m_0 < \infty$, and \mathcal{F}_{m_0-1} measurable nonnegative random value $\varphi_{m_0}^s$, $\hat{P}_1(\varphi_{m_0}^s > 0) > 0$, such that the inequalities there hold

$$f_{m_0-1} - E^{\hat{P}_i}\{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^s, \quad i = \overline{1, s},$$

and for a maximal element $g^s = \{g_m^s\}_{m=0}^\infty$ in a certain maximal chain $\tilde{G}_s \subseteq G_s$ there hold equalities (2.34), (2.35). Equalities (2.35) yield the equalities

$$\begin{aligned}
E^Q\{g_m^s - g_{m-1}^s | \mathcal{F}_{m-1}\} &= \\
\frac{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i}\{g_m^s - g_{m-1}^s | \mathcal{F}_{m-1}\}}{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\}} &= \\
\frac{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i}\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}}{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\}} &= E^Q\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}, \quad (2.36)
\end{aligned}$$

$$m = \overline{1, \infty}, \quad Q \in \hat{M}_s.$$

where $\hat{\varphi}_i = \frac{d\hat{P}_i}{dP_1}$, $i = \overline{1, n}$. Taking into account the equalities (2.34), we obtain

$$\begin{aligned}
E^Q\{f_m + g_m^s | \mathcal{F}_{m-1}\} &= \\
\frac{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\} E^{\hat{P}_i}\{f_m + g_m^s | \mathcal{F}_{m-1}\}}{\sum_{i=1}^s \gamma_i E^{\hat{P}_1}\{\hat{\varphi}_i | \mathcal{F}_{m-1}\}} &=
\end{aligned}$$

$$f_{m-1} + g_{m-1}^s, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.37)$$

Thus, we have

$$E^Q\{g_m^s - g_{m-1}^s | \mathcal{F}_{m-1}\} = E^Q\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.38)$$

$$E^Q\{f_m + g_m^s | \mathcal{F}_{m-1}\} = f_{m-1} + g_{m-1}^s, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.39)$$

Let us introduce into consideration a random process $\{N_m, \mathcal{F}_m\}_{m=0}^\infty$, where

$$N_0 = f_0, \quad N_m = f_m + \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{1, \infty}.$$

It is evident that $E^Q|N_m| < \infty$, $m = \overline{1, \infty}$, $Q \in \hat{M}_s$. The definition of $\{N_m, \mathcal{F}_m\}_{m=0}^\infty$ and the formulae (2.38), (2.39) yield

$$\begin{aligned}
E^Q\{N_{m-1} - N_m | \mathcal{F}_{m-1}\} &= E^Q\{f_{m-1} - f_m - \bar{g}_m^0 | \mathcal{F}_{m-1}\} = \\
&= E^Q\{g_m^s - g_{m-1}^s - \bar{g}_m^0 | \mathcal{F}_{m-1}\} = 0, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s.
\end{aligned}$$

The last equalities imply

$$E^Q\{N_m|\mathcal{F}_{m-1}\} = N_{m-1}, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s.$$

Due to arbitrariness of the set of measures $\hat{P}_1, \dots, \hat{P}_s, \hat{P}_i \in \tilde{M}_n$, we have

$$E^P\{N_m|\mathcal{F}_{m-1}\} = N_{m-1}, \quad P \in \tilde{M}_n, \quad m = \overline{1, \infty}. \quad (2.40)$$

So, the set G_0 of adapted non-decreasing processes $\{g_m\}_{m=0}^\infty$, $g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from \tilde{M}_n contains nonzero element $\tilde{g}^0 = \{\tilde{g}_m^0\}_{m=0}^\infty$, $\tilde{g}_0^0 = 0$, $\tilde{g}_m^0 = \sum_{i=1}^m \tilde{g}_i^0$, $m = \overline{1, \infty}$, which is a maximal element in a maximal chain \tilde{G}_0 containing this element. Really, if $g^0 = \{g_m^0\}_{m=0}^\infty$, $g_0^0 = 0$, is a maximal element in the maximal chain $\tilde{G}_0 \subseteq G_0$, then there hold inequalities

$$E^{P_0}\{f_m + g_m^0|\mathcal{F}_k\} \leq f_k + g_k^0, \quad m = \overline{1, \infty}, \quad 1 \leq k \leq m, \quad P_0 \in \tilde{M}_n, \quad (2.41)$$

$$E^{P_0}(f_m + g_m^0) \leq f_0, \quad m = \overline{1, \infty}, \quad P_0 \in \tilde{M}_n. \quad (2.42)$$

and inequality $\tilde{g}^0 \preceq g^0$ meaning that $\tilde{g}_m^0 \leq g_m^0$, $m = \overline{0, \infty}$. Equalities (2.40) yield

$$E^{P_0}(f_m + \tilde{g}_m^0) = f_0, \quad m = \overline{1, \infty}, \quad P_0 \in \tilde{M}_n. \quad (2.43)$$

Inequalities (2.42) and equalities (2.43) imply

$$f_0 \geq E^{P_0}(f_m + g_m^0) \geq E^{P_0}(f_m + \tilde{g}_m^0) = f_0, \quad m = \overline{1, \infty}, \quad P_0 \in \tilde{M}_n. \quad (2.44)$$

The last inequalities lead to equalities

$$E^{P_0}(g_m^0 - \tilde{g}_m^0) = 0, \quad m = \overline{1, \infty}, \quad P_0 \in \tilde{M}_n. \quad (2.45)$$

But

$$g_m^0 - \tilde{g}_m^0 \geq 0, \quad m = \overline{0, \infty}. \quad (2.46)$$

The equalities (2.45) and inequalities (2.46) yield $g_m^0 = \tilde{g}_m^0$, $m = \overline{0, \infty}$, or $\tilde{g}^0 = g^0$.

Prove that $G_n = G_0$, where G_n is a set of non-decreasing processes $g = \{g_m\}_{m=0}^\infty$ such that $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from \tilde{M}_n . Really, if $g = \{g_m\}_{m=0}^\infty$ is a non-decreasing process from G_n , then it belongs to G_0 , owing to that $M_n \supset \tilde{M}_n$ and $G_n \subseteq G_0$. Suppose that $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, is a non-decreasing process from G_0 . It means that

$$E^Q\{f_m + g_m|\mathcal{F}_k\} \leq f_k + g_k, \quad m = \overline{1, \infty}, \quad 0 \leq k \leq m, \quad Q \in \tilde{M}_n. \quad (2.47)$$

The last inequalities can be written in the form

$$\sum_{i=1}^n \alpha_i \int_A (f_m + g_m) dP_i \leq \sum_{i=1}^n \alpha_i \int_A (f_k + g_k) dP_i, \quad m = \overline{1, \infty}, \quad 0 \leq k \leq m,$$

$$A \in \mathcal{F}_k, \quad \alpha_i > 0, \quad i = \overline{1, n}.$$

By passing to the limit, as $\alpha_j \rightarrow 0$, $\alpha_j > 0$, $j \neq i$, $\alpha_i \rightarrow 1$, we obtain

$$\int_A (f_m + g_m) dP_i \leq \int_A (f_k + g_k) dP_i, \quad i = \overline{1, n}, \quad A \in \mathcal{F}_k, \quad m = \overline{1, \infty}, \quad 0 \leq k \leq m.$$

The last inequalities yield inequalities

$$\sum_{i=1}^n \alpha_i \int_A (f_m + g_m) dP_i \leq \sum_{i=1}^n \alpha_i \int_A (f_k + g_k) dP_i, \quad m = \overline{1, \infty}, \quad 0 \leq k \leq m,$$

$$A \in \mathcal{F}_k, \quad \alpha_i \geq 0, \quad i = \overline{1, n},$$

or

$$E^Q\{f_m + g_m | \mathcal{F}_k\} \leq f_k + g_k, \quad m = \overline{1, \infty}, \quad 0 \leq k \leq m, \quad Q \in M_n.$$

It means that $g = \{g_m\}_{m=0}^\infty$ belongs to G_n . On the basis of the above proved, for the maximal element $\tilde{g}^0 = \{\tilde{g}_m^0\}_{m=0}^\infty$ in the maximal chain $\tilde{G}_0 \subseteq G_0$ the equalities

$$E^Q\{f_m + \tilde{g}_m^0 | \mathcal{F}_k\} = f_k + \tilde{g}_k^0, \quad m = \overline{1, \infty}, \quad 1 \leq k \leq m, \quad Q \in \tilde{M}_n, \quad (2.48)$$

$$E^Q(f_m + \tilde{g}_m^0) = f_0, \quad m = \overline{1, \infty}, \quad Q \in \tilde{M}_n, \quad (2.49)$$

are valid. From proved equality $G_n = G_0$, it follows that \tilde{G}_0 is a maximal chain in G_n .

As far as, G_0 coincides with G_n we proved that the maximal element \tilde{g}^0 in a certain maximal chain in G_n satisfies equalities

$$E^{P_0}\{f_m + \tilde{g}_m^0 | \mathcal{F}_k\} = f_k + \tilde{g}_k^0, \quad m = \overline{1, \infty}, \quad 1 \leq k \leq m, \quad P_0 \in M_n, \quad (2.50)$$

$$E^{P_0}(f_m + \tilde{g}_m^0) = f_0, \quad m = \overline{1, \infty}, \quad P_0 \in M_n. \quad (2.51)$$

Due to arbitrariness of the set of measure P_1, \dots, P_n , $P_i \in M$, the set G contains nonzero element \tilde{g}^0 and in the maximal chain $\tilde{G} \subseteq G$ containing element \tilde{g}^0 the maximal element $g^0 = \{g_m^0\}_{m=0}^\infty$, $g_0^0 = 0$, coincides with \tilde{g}^0 . The last statement can be proved as in the case of maximal chain \tilde{G}_0 . So,

$$E^{P_0}\{f_m + g_m^0 | \mathcal{F}_k\} = f_k + g_k^0, \quad m = \overline{1, \infty}, \quad 1 \leq k \leq m, \quad P_0 \in M, \quad (2.52)$$

$$E^{P_0}(f_m + g_m^0) = f_0, \quad m = \overline{1, \infty}, \quad P_0 \in M. \quad (2.53)$$

Denote by $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ a martingale relative to the set of measures M , where $\bar{M}_m = f_m + g_m^0$, $m = \overline{1, \infty}$. Due to Theorem 2.3 conditions, the supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ and non-decreasing process $g^0 = \{g_m^0\}_{m=0}^\infty$ are uniformly integrable relative to any measure from M , since for the non-decreasing process $g^0 = \{g_m^0\}_{m=0}^\infty$ there hold bounds $E^P g_m^0 < T + f_0$, $m = \overline{1, \infty}$, $P \in M$. Therefore, the martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ is uniformly integrable relative to any measure from M . So, with probability 1 relative to every measure from M there exist limits

$$\lim_{m \rightarrow \infty} \bar{M}_m = M_\infty = f_\infty + g_\infty^0, \quad \lim_{m \rightarrow \infty} f_m = f_\infty, \quad \lim_{m \rightarrow \infty} g_m^0 = g_\infty^0.$$

Moreover, the representation

$$\bar{M}_m = E^P\{(f_\infty + g_\infty^0)|\mathcal{F}_m\}, \quad m = \overline{1, \infty}, \quad P \in M, \quad (2.54)$$

holds, where $\bar{M} = \{\bar{M}_m\}_{m=0}^\infty$ does not depend on $P \in M$.

In the next theorem we give the necessary and sufficient conditions of regularity of supermartingales.

Theorem 2.4 *Let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M satisfy conditions (2.4). The necessary and sufficient conditions for it to be a regular one is the existence of adapted nonnegative random process $\bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^\infty$, $\bar{g}_0^0 = 0$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, $P \in M$, such that equalities*

$$E^P\{f_{m-1} - f_m|\mathcal{F}_{m-1}\} = E^P\{\bar{g}_m^0|\mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad P \in M, \quad (2.55)$$

are valid.

Proof. Necessity. If $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a regular supermartingale, then there exist a martingale $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ and a non-decreasing nonnegative random process $\{g_m, \mathcal{F}_m\}_{m=0}^\infty$, $g_0 = 0$, such that

$$f_m = \bar{M}_m - g_m, \quad m = \overline{1, \infty}. \quad (2.56)$$

As before, equalities (2.56) yield inequalities $E^P g_m \leq f_0 + T$, $m = \overline{1, \infty}$, and equalities

$$\begin{aligned} E^P\{f_{m-1} - f_m|\mathcal{F}_{m-1}\} &= \\ &= E^P\{g_m - g_{m-1}|\mathcal{F}_{m-1}\} = E^P\{\bar{g}_m^0|\mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad P \in M, \end{aligned} \quad (2.57)$$

where we introduced the denotation $\bar{g}_m^0 = g_m - g_{m-1} \geq 0$. It is evident that $E^P \bar{g}_m^0 \leq 2(f_0 + T)$.

Sufficiency. If there exists an adapted nonnegative random process $\bar{g}^0 = \{\bar{g}_m^0\}_{m=0}^\infty$, $\bar{g}_0^0 = 0$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, such that the equalities (2.55) are valid, then let us consider a random process $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$, where

$$\bar{M}_0 = f_0, \quad \bar{M}_m = f_m + \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{1, \infty}. \quad (2.58)$$

It is evident that $E^P|\bar{M}_m| < \infty$ and

$$E^P\{\bar{M}_{m-1} - \bar{M}_m|\mathcal{F}_{m-1}\} = E^P\{f_{m-1} - f_m - \bar{g}_m^0|\mathcal{F}_{m-1}\} = 0.$$

Theorem 2.4 is proven.

In the next Theorem we describe the structure of non-decreasing process for a regular supermartingale.

Theorem 2.5 *Let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M satisfy conditions (2.4). The necessary and sufficient conditions for it to be regular one is the existence of a non-decreasing adapted process $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, and adapted processes $\bar{\Psi}^j = \{\bar{\Psi}_m^j\}_{m=0}^\infty$, $\bar{\Psi}_0^j = 0$, $j = \overline{1, n}$, such that between elements g_m , $m = \overline{1, \infty}$, of non-decreasing process $g = \{g_m\}_{m=0}^\infty$ the relations*

$$g_m - g_{m-1} = f_{m-1} - E^{P_j}\{f_m|\mathcal{F}_{m-1}\} + \bar{\Psi}_m^j, \quad m = \overline{1, \infty}, \quad j = \overline{1, n}, \quad (2.59)$$

are valid for each set of measures $P_1, \dots, P_n \in M$, where $E^{P_j}|\bar{\Psi}_m^j| < \infty$, $E^{P_j}\{\bar{\Psi}_m^j|\mathcal{F}_{m-1}\} = 0$, $j = \overline{1, n}$, $m = \overline{1, \infty}$.

Proof. The necessity. Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a regular supermartingale. Then for it the representation

$$f_m + g_m = M_m, \quad m = \overline{1, \infty}, \quad j = \overline{1, n}, \quad (2.60)$$

is valid, where $\{g_m\}_{m=0}^\infty$, $g_0 = 0$, is a non-decreasing adapted process, $\{M_m, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to the set of measures M . For any finite set of measures $P_1, \dots, P_n \in M$, we have

$$E^{P_j}\{f_m + g_m | \mathcal{F}_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = \overline{1, \infty}, \quad j = \overline{1, n}. \quad (2.61)$$

Hence, we have

$$\begin{aligned} E^{P_j}\{g_m - g_{m-1} | \mathcal{F}_{m-1}\} = \\ f_{m-1} - E^{P_j}\{f_m | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad j = \overline{1, n}. \end{aligned} \quad (2.62)$$

Let us put

$$\bar{\Psi}_m^j = g_m - g_{m-1} - E^{P_j}\{g_m - g_{m-1} | \mathcal{F}_{m-1}\}. \quad (2.63)$$

The assumptions of Theorem 2.5 and Lemma 2.3, the representation (2.63) imply $E^{P_j}|\bar{\Psi}_m^j| < 4(f_0 + T)$, $E^{P_j}\{\bar{\Psi}_m^j | \mathcal{F}_{m-1}\} = 0$, $j = \overline{1, n}$, $m = \overline{1, \infty}$. This proves the necessity.

The sufficiency. For any set of measures $P_1, \dots, P_n \in M$ the representation (2.59) for a non-decreasing adapted process $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, is valid. Hence, we obtain (2.62) and (2.61). The equalities (2.62), (2.61) and the formula

$$\begin{aligned} E^P\{f_m + g_m | \mathcal{F}_{m-1}\} = \frac{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_{m-1}\} E^{P_i}\{f_m + g_m | \mathcal{F}_{m-1}\}}{\sum_{i=1}^n \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_{m-1}\}}, \quad P \in M_n, \\ \varphi_i = \frac{dP_i}{dP_1}, \quad i = \overline{1, n}, \end{aligned}$$

imply

$$E^P\{f_m + g_m | \mathcal{F}_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = \overline{1, \infty}, \quad P \in M_n.$$

Arbitrariness of the set of measures $P_1, \dots, P_n \in M$ and fulfilment of the condition (2.4) for the supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ imply its regularity.

Further, we consider a class of supermartingales F satisfying conditions

$$\sup_{P \in M} E^P|f_m| < \infty, \quad m = \overline{0, \infty}.$$

Definition 2.4 A supermartingale $f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty \in F$ is said to be local regular one if there exists an increasing sequence of nonrandom stopping times $\tau_{k_s} = k_s$, $k_s < \infty$, $s = \overline{1, \infty}$, $\lim_{s \rightarrow \infty} k_s = \infty$, such that the stopped process $f^{\tau_{k_s}} = \{f_{m \wedge \tau_{k_s}}, \mathcal{F}_m\}_{m=0}^\infty$ is a regular supermartingale for every $\tau_{k_s} = k_s$, $k_s < \infty$, $s = \overline{1, \infty}$.

Theorem 2.6 Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a supermartingale relative to a convex set of equivalent measures M , belonging to the class F , for which the representation

$$f_m = M_m - g_m^0, \quad m = \overline{0, \infty}, \quad (2.64)$$

is valid, where $\{M_m\}_{m=0}^\infty$ is a martingale relative to a convex set of equivalent measures M such that

$$E^P |M_m| < \infty, \quad m = \overline{0, \infty}, \quad P \in M,$$

$g^0 = \{g_m^0\}_{m=0}^\infty$, $g_0^0 = 0$, is a non-decreasing adapted process. Then $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular supermartingale.

Proof. The representation (2.64) and assumptions of Theorem 2.6 imply inequalities $E^P g_m^0 < \infty$, $m = \overline{1, \infty}$, $P \in M$. For any measure $P \in M$, therefore we have

$$E^P \{f_m + g_m^0 | \mathcal{F}_{m-1}\} = M_{m-1} = f_{m-1} + g_{m-1}^0, \quad m = \overline{1, \infty}. \quad (2.65)$$

Consider a sequence of stopping times $\tau_s = s$, $s = \overline{1, \infty}$. Equalities (2.65) yield

$$E^P \{f_{m \wedge \tau_s} + g_{m \wedge \tau_s}^0 | \mathcal{F}_{m-1}\} = M_{(m-1) \wedge \tau_s} = f_{(m-1) \wedge \tau_s} + g_{(m-1) \wedge \tau_s}^0, \quad (2.66)$$

$$m = \overline{1, \infty}, \quad P \in M.$$

For the stopped supermartingale $\{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^\infty$, the set G of adapted non-decreasing processes $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, such that $\{f_{m \wedge \tau_s} + g_m, \mathcal{F}_m\}_{m=0}^\infty$ is a supermartingale relative to a convex set of equivalent measures M contains nonzero element $g^{0, \tau_s} = \{g_{m \wedge \tau_s}^0\}_{m=0}^\infty$, $g_0^0 = 0$. Consider a maximal chain $\tilde{G} \subseteq G$ containing this element and let $g = \{g_m\}_{m=0}^\infty$, $g_0 = 0$, be a maximal element in \tilde{G} which exists, since the stopped supermartingale $\{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^\infty$ is such that $|f_{m \wedge \tau_s}| \leq \sum_{i=0}^s |f_i| = \varphi$, $m = \overline{0, \infty}$, $E^P \varphi \leq \sum_{i=0}^s \sup_{P \in M} E^P |f_i| = T < \infty$. Then

$$E^P \{f_{m \wedge \tau_s} + g_m | \mathcal{F}_{m-1}\} \leq f_{(m-1) \wedge \tau_s} + g_{m-1}, \quad m = \overline{1, \infty}. \quad (2.67)$$

Equalities (2.66) and inequality $g^{0, \tau_s} \preceq g$ imply

$$f_0 = E^P \{f_{m \wedge \tau_s} + g_{m \wedge \tau_s}^0\} \leq E^P \{f_{m \wedge \tau_s} + g_m\} \leq f_0, \quad m = \overline{1, \infty}, \quad P \in M. \quad (2.68)$$

The last inequalities yield

$$E^P \{f_{m \wedge \tau_s} + g_m\} = f_0, \quad m = \overline{1, \infty}, \quad P \in M. \quad (2.69)$$

The equalities (2.69), inequality $g^{0, \tau_s} \preceq g$, and equalities

$$E^P \{f_{m \wedge \tau_s} + g_{m \wedge \tau_s}^0\} = M_0 = f_0, \quad m = \overline{1, \infty}, \quad P \in M, \quad (2.70)$$

imply that $g^{0, \tau_s} = g$.

So, we proved that the stopped supermartingale $\{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^\infty$ is regular one for every stopping time τ_s , $s = \overline{1, \infty}$, converging to the infinity, as $s \rightarrow \infty$. This proves Theorem 2.6.

Theorem 2.7 *On a measurable space $\{\Omega, \mathcal{F}\}$, let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M belongs to a class F and there exists a nonnegative adapted random process $\{\bar{g}_m^0\}_{m=1}^\infty$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, $P \in M$, such that*

$$f_{m-1} - E^P\{f_m | \mathcal{F}_{m-1}\} = E^P\{\bar{g}_m^0 | \mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}, \quad P \in M, \quad (2.71)$$

then $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular supermartingale.

Proof. To prove Theorem 2.7 let us consider a random process

$$\bar{M}_m = f_m + \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{1, \infty}, \quad P \in M, \quad f_0 = \bar{M}_0.$$

It is evident that $E^P |\bar{M}_m| < \infty$, $m = \overline{1, \infty}$, $P \in M$, and $E^P\{\bar{M}_m | \mathcal{F}_{m-1}\} = \bar{M}_{m-1}$, $m = \overline{1, \infty}$, $P \in M$. Therefore, for f_m the representation

$$f_m = \bar{M}_m - g_m, \quad m = \overline{0, \infty}, \quad (2.72)$$

is valid, where $g_m = \sum_{i=1}^m \bar{g}_i^0$. Supermartingale (2.72) satisfies conditions of the Theorem 6. The Theorem 2.7 is proved.

3 Description of local regular supermartingales.

Below, we describe local regular supermartingales. For this we need some auxiliary statements.

Let P_1, \dots, P_k be a family of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$ and let us introduce denotation M for a convex set of equivalent measures

$$M = \left\{ Q, \quad Q = \sum_{i=1}^k \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = \overline{1, k}, \quad \sum_{i=1}^k \alpha_i = 1 \right\}.$$

Lemma 3.1 *If ξ is an integrable random value relative to the set of equivalent measures P_1, \dots, P_k , then the formula*

$$\text{ess sup}_{Q \in M} E^Q\{\xi | \mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_i}\{\xi | \mathcal{F}_n\} \quad (3.1)$$

is valid almost everywhere relative to the measure P_1 .

Proof. The definition for ess sup of non countable family of random variable see [2]. Using the formula

$$E^Q\{\xi | \mathcal{F}_n\} = \frac{\sum_{i=1}^k \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_n\} E^{P_i}\{\xi | \mathcal{F}_n\}}{\sum_{i=1}^k \alpha_i E^{P_i}\{\varphi_i | \mathcal{F}_n\}}, \quad Q \in M, \quad (3.2)$$

where $\varphi_i = \frac{dP_i}{dP_1}$, we obtain the inequality

$$E^Q\{\xi | \mathcal{F}_n\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi | \mathcal{F}_n\}, \quad Q \in M,$$

or,

$$\operatorname{ess\,sup}_{Q \in M} E^Q\{\xi|\mathcal{F}_n\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}.$$

On the other side [2],

$$E^{P_i}\{\xi|\mathcal{F}_n\} \leq \operatorname{ess\,sup}_{Q \in M} E^Q\{\xi|\mathcal{F}_n\}, \quad i = \overline{1, k}.$$

Therefore,

$$\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\} \leq \operatorname{ess\,sup}_{Q \in M} E^Q\{\xi|\mathcal{F}_n\}.$$

The Lemma 3.1 is proved.

Lemma 3.2 *Let G be a sub σ -algebra of σ -algebra \mathcal{F} and $f_s, s \in S$, be a nonnegative bounded family of random values relative to every measure from M . Then*

$$E^P\{\operatorname{ess\,sup}_{s \in S} f_s|G\} \geq \operatorname{ess\,sup}_{s \in S} E^P\{f_s|G\}, \quad P \in M. \quad (3.3)$$

Proof. From the definition of $\operatorname{ess\,sup}$ [2], we have the inequalities

$$\operatorname{ess\,sup}_{s \in S} f_s \geq f_t, \quad t \in S. \quad (3.4)$$

Therefore,

$$E^P\{\operatorname{ess\,sup}_{s \in S} f_s|G\} \geq E^P\{f_t|G\}, \quad t \in S. \quad (3.5)$$

The last implies

$$E^P\{\operatorname{ess\,sup}_{s \in S} f_s|G\} \geq \operatorname{ess\,sup}_{s \in S} E^P\{f_s|G\}. \quad (3.6)$$

In the next Lemma, we present formula for calculation of conditional expectation relative to another measure from M .

Lemma 3.3 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let M be a convex set of equivalent measures and let ξ be a bounded random value. Then the following formulas*

$$E^{P_1}\{\xi|\mathcal{F}_n\} = E^{P_2}\{\xi\varphi_n^{P_1}|\mathcal{F}_n\}, \quad n = \overline{1, \infty}, \quad (3.7)$$

are valid, where

$$\varphi_n^{P_1} = \frac{dP_1}{dP_2} \left[E^{P_2} \left\{ \frac{dP_1}{dP_2} | \mathcal{F}_n \right\} \right]^{-1}, \quad P_1, P_2 \in M.$$

Proof. The proof of the Lemma 3.3 is evident.

Lemma 3.4 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let ξ be a non-negative bounded random value. Then the formulas*

$$E^Q\{\operatorname{ess\,sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} = \operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_m\}, \quad Q \in M, \quad n > m, \quad (3.8)$$

are valid, where

$$\varphi_n^P = \frac{dP}{dQ} \left[E^Q \left\{ \frac{dP}{dQ} | \mathcal{F}_n \right\} \right]^{-1}, \quad P \in M.$$

Proof. From the Lemma 3.3, we obtain

$$\operatorname{ess\,sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} = \operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}, \quad Q \in M.$$

Due to Lemma 3.2, we obtain the inequality

$$\begin{aligned} E^Q\{\operatorname{ess\,sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} &= E^Q\{\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}|\mathcal{F}_m\} \geq \\ &\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_m\}. \end{aligned}$$

Let us prove reciprocal inequality

$$E^Q\{\operatorname{ess\,sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_m\}.$$

From the definition of $\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}$, there exists a countable set $D = \{\bar{P}_i \in M, i = \overline{1, \infty}\}$ [2] such that

$$\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\} = \sup_{P \in D} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}.$$

The sequence $\varphi_k = \sup_{P \in D} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\} - \max_{1 \leq i \leq k} E^Q\{\xi\varphi_n^{\bar{P}_i}|\mathcal{F}_n\}$, $k = \overline{1, \infty}$, converges to zero with probability one, as k tends to infinity. It is evident that

$$\max_{1 \leq i \leq k} E^Q\{\xi\varphi_n^{\bar{P}_i}|\mathcal{F}_n\} = E^Q\{\xi\varphi_n^{\bar{P}_{\tau_k}}|\mathcal{F}_n\},$$

where

$$\begin{aligned} \tau_1 &= 1, \\ \tau_i &= \begin{cases} \tau_{i-1}, & E^Q\{\xi\varphi_n^{\bar{P}_{\tau_{i-1}}}|\mathcal{F}_n\} > E^Q\{\xi\varphi_n^{\bar{P}_i}|\mathcal{F}_n\}, \\ i, & E^Q\{\xi\varphi_n^{\bar{P}_i}|\mathcal{F}_n\} \geq E^Q\{\xi\varphi_n^{\bar{P}_{\tau_{i-1}}}|\mathcal{F}_n\}, \end{cases} \quad i = \overline{2, \infty}. \end{aligned}$$

Therefore,

$$\begin{aligned} E^Q\{\operatorname{ess\,sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} &= E^Q\{\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}|\mathcal{F}_m\} = \\ &= E^Q\{\sup_{P \in D} E^Q\{\xi\varphi_n^P|\mathcal{F}_n\}|\mathcal{F}_m\} = E^Q\{\lim_{k \rightarrow \infty} \max_{1 \leq i \leq k} E^Q\{\xi\varphi_n^{\bar{P}_i}|\mathcal{F}_n\}|\mathcal{F}_m\} = \\ &\lim_{k \rightarrow \infty} E^Q\{E^Q\{\xi \max_{1 \leq i \leq k} \varphi_n^{\bar{P}_i}|\mathcal{F}_n\}|\mathcal{F}_m\} = \lim_{k \rightarrow \infty} E^Q\{\xi\varphi_n^{\bar{P}_{\tau_k}}|\mathcal{F}_m\} \leq \\ &\operatorname{ess\,sup}_{P \in M} E^Q\{\xi\varphi_n^P|\mathcal{F}_m\}. \end{aligned}$$

In equalities above, we can change the limits under conditional expectation sign, since with probability one the inequalities

$$\max_{1 \leq i \leq k} \varphi_n^{\bar{P}_i} \leq \max_{1 \leq i \leq k+1} \varphi_n^{\bar{P}_i}, \quad k = \overline{1, \infty},$$

are valid. Lemma 3.4 is proved.

The next Lemma is proved, as Lemma 3.4.

Lemma 3.5 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let ξ be a non-negative bounded random value. Then the equalities*

$$E^Q\{\xi \operatorname{ess\,sup}_{P \in M} \varphi_n^P | \mathcal{F}_n\} = \operatorname{ess\,sup}_{P \in M} E^Q\{\xi \varphi_n^P | \mathcal{F}_n\}, \quad Q \in M, \quad n = \overline{0, \infty}, \quad (3.9)$$

are valid, where

$$\varphi_n^P = \frac{dP}{dQ} \left[E^Q \left\{ \frac{dP}{dQ} | \mathcal{F}_n \right\} \right]^{-1}.$$

Lemma 3.6 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, for every non-negative bounded random value ξ the inequalities*

$$E^Q\{\operatorname{ess\,sup}_{P \in M} E^P\{\xi | \mathcal{F}_n\} | \mathcal{F}_m\} \leq \operatorname{ess\,sup}_{P \in M} E^P\{\xi | \mathcal{F}_m\}, \quad n > m, \quad (3.10)$$

are valid.

Proof. From the Lemma 3.4, we have

$$\begin{aligned} E^Q\{\sup_{P \in D} E^P\{\xi | \mathcal{F}_n\} | \mathcal{F}_m\} &= E^Q\{\sup_{P \in D} E^Q\{\xi \varphi_n^P | \mathcal{F}_n\} | \mathcal{F}_m\} = \\ &= \sup_{P \in D} E^Q\{\xi \varphi_n^P | \mathcal{F}_m\}, \quad n > m, \end{aligned} \quad (3.11)$$

where D is a countable subset of the set M . Without loss of generality, we assume that the set of measures $\{P_1, \dots, P_k\}$ belongs to the countable set $D = \{\bar{P}_i \in M, i = \overline{1, \infty}\}$. First, we assume that $Q \in D$. Then, it is evident that the following equalities

$$\bigcup_{i=1}^{\infty} \left\{ \omega, E^Q \left\{ \frac{d\bar{P}_i}{dQ} | \mathcal{F}_n \right\} \geq E^Q \left\{ \frac{d\bar{P}_i}{dQ} | \mathcal{F}_m \right\} \right\} = \Omega, \quad n > m, \quad (3.12)$$

are valid. Due to (3.12), for every $\omega \in \Omega$ there exist $1 \leq i < \infty$ such that

$$\frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_n\}} \leq \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_m\}}. \quad (3.13)$$

Therefore,

$$\sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_n\}} \leq \sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_m\}}. \quad (3.14)$$

From (3.14), we obtain the inequality

$$E^Q\{\sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_n\}} | \mathcal{F}_m\} \leq E^Q\{\sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_m\}} | \mathcal{F}_m\}. \quad (3.15)$$

Or,

$$E^Q\{E^Q\{\sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_n\}} | \mathcal{F}_n\} | \mathcal{F}_m\} \leq E^Q\{\sup_{\bar{P}_i \in D} \frac{\xi \frac{d\bar{P}_i}{dQ}}{E^Q\{\frac{d\bar{P}_i}{dQ} | \mathcal{F}_m\}} | \mathcal{F}_m\}. \quad (3.16)$$

The Lemmas 3.4, 3.5 and inequality (3.16) prove Lemma 3.6, as $Q \in D$. Let $Q \in M$. Since the set of measures $\{P_1, \dots, P_k\}$ belongs to D we complete the proof of the Lemma 3.6, using the formula

$$E^Q\{\Phi|\mathcal{F}_m\} = \frac{\sum_{i=1}^k \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_m\} E^{P_i}\{\Phi|\mathcal{F}_m\}}{\sum_{i=1}^k \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_m\}}, \quad Q \in M, \quad (3.17)$$

and proved above inequalities, as $Q \in D$, where $\Phi = \sup_{\bar{P}_i \in D} E^{\bar{P}_i}\{\xi|\mathcal{F}_n\}$, $\varphi_i = \frac{dP_i}{dP_1}$, $i = \overline{1, k}$. The Lemma 3.6 is proved.

Lemma 3.7 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let ξ be an integrable relative to the set of equivalent measures P_1, \dots, P_k random value. Then the inequalities*

$$E^Q\left\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\right\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_m\}, \quad n > m, \quad Q \in M, \quad (3.18)$$

are valid.

Proof. Using the Lemma 3.1 and the Lemma 3.6 for a bounded ξ , we prove the Lemma 3.7 inequalities (3.18). Let us consider the case, as $\max_{1 \leq i \leq k} E^{P_i}\xi < \infty$. Let $\xi_s, s = \overline{1, \infty}$, be a sequence of bounded random values converging to ξ monotonously. Then

$$E^Q\left\{\max_{1 \leq i \leq k} E^{P_i}\{\xi_s|\mathcal{F}_n\}|\mathcal{F}_m\right\} \leq \max_{1 \leq i \leq k} E^Q\{\xi_s|\mathcal{F}_m\}, \quad l = \overline{1, k}. \quad (3.19)$$

Due to monotony convergence of ξ_s to ξ , as $s \rightarrow \infty$, we can pass to the limit under conditional expectations on the left and on the right in inequalities (3.19) that proves the Lemma 3.7.

Lemma 3.8 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let ξ be a nonnegative integrable random value with respect to a set of equivalent measures $\{P_1, \dots, P_k\}$ and such that*

$$E^{P_i}\xi = M_0, \quad i = \overline{1, k}, \quad (3.20)$$

then the random process $\{M_m = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to a convex set of equivalent measures M .

Proof. Due to Lemma 3.7, a random process $\{M_m = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a supermartingale, that is,

$$E^P\{M_m|\mathcal{F}_{m-1}\} \leq M_{m-1}, \quad m = \overline{1, \infty}, \quad P \in M.$$

Or, $E^P M_m \leq M_0$. From the other side,

$$E^{P_s}[\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_m\}] \geq \max_{1 \leq i \leq k} E^{P_s} E^{P_i}\{\xi|\mathcal{F}_m\} \geq M_0, \quad s = \overline{1, k}.$$

The above inequalities imply $E^{P_s} M_m = M_0$, $m = \overline{1, \infty}$, $s = \overline{1, k}$. The last equalities lead to equalities $E^P M_m = M_0$, $m = \overline{1, \infty}$, $P \in M$. The fact that M_m is a supermartingale relative to the set of measures M and the above equalities prove the Lemma 3.8, since the Lemma 2.1 conditions are valid.

Theorem 3.1 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let ξ be a \mathcal{F}_N -measurable nonnegative integrable relative to a set of equivalent measures $\{P_1, \dots, P_k\}$ random value, $N < \infty$. Then a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$, where*

$$f_m = \operatorname{ess\,sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \quad m = \overline{1, \infty}, \quad \max_{1 \leq i \leq k} E^{P_i} \xi < \infty, \quad (3.21)$$

is local regular one if and only if

$$E^{P_i} \xi = f_0, \quad i = \overline{1, k}. \quad (3.22)$$

Proof. The necessity. Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a local regular supermartingale. Then there exists a sequence of nonrandom stopping times $\tau_s = n_s$, $s = \overline{1, \infty}$, such that for every n_s there exists $\varphi = \sum_{m=1}^{n_s} \sum_{i=1}^k E^{P_i} \{\xi | \mathcal{F}_m\}$ satisfying inequalities

$$\max_{1 \leq j \leq k} E^{P_j} \varphi \leq \sum_{m=1}^{n_s} \sum_{i=1}^k \max_{1 \leq j \leq k} E^{P_j} E^{P_i} \{\xi | \mathcal{F}_m\} \leq$$

$$\sum_{m=1}^{n_s} \sum_{i=1}^k \max_{1 \leq j \leq k} E^{P_j} \max_{1 \leq i \leq k} E^{P_i} \{\xi | \mathcal{F}_m\} \leq$$

$$\sum_{m=1}^{n_s} \sum_{i=1}^k \max_{1 \leq j \leq k} E^{P_j} \max_{1 \leq i \leq k} E^{P_i} \xi = n_s k \max_{1 \leq i \leq k} E^{P_i} \xi,$$

$$\sup_{P \in M} E^P \varphi \leq \max_{1 \leq j \leq k} E^{P_j} \varphi \leq n_s k \max_{1 \leq i \leq k} E^{P_i} \xi,$$

and nonnegative adapted random process $\{\bar{g}_m^0\}_{m=0}^\infty$, $\bar{g}_0^0 = 0$, $E^{P_i} \bar{g}_m^0 < \infty$, $0 \leq m \leq n_s$ such that

$$f_m + \sum_{i=1}^m \bar{g}_i^0 = \bar{M}_m, \quad E^P \bar{M}_m = f_0, \quad 0 \leq m \leq n_s, \quad P \in M.$$

If $n_s > N$, then

$$E^{P_i}(\xi + \sum_{i=1}^N \bar{g}_i^0) = E^{P_i} \xi + E^{P_i} \sum_{i=1}^N \bar{g}_i^0 = f_0.$$

But there exists $1 \leq i_1 \leq k$ such that $E^{P_{i_1}} \xi = f_0$. Therefore, $E^{P_{i_1}} \sum_{i=1}^N \bar{g}_i^0 = 0$. Due to equivalence of measures P_i , $i = \overline{1, k}$, we obtain

$$E^{P_i} \xi = f_0, \quad i = \overline{1, k}, \quad (3.23)$$

where $f_0 = \sup_{P \in M} E^P \xi$.

Sufficiency. If conditions (3.23) are satisfied, then $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale, where $\bar{M}_m = \sup_{P \in M} E^P \{\xi | \mathcal{F}_m\}$. The last implies local regularity of $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$.

The Theorem 3.1 is proved.

Below we consider an arbitrary convex set of equivalent measures M on a measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it. Introduce into consideration a set A_0 of all integrable nonnegative random values ξ relative to a convex set of equivalent measures M satisfying conditions

$$E^P \xi = 1, \quad P \in M. \quad (3.24)$$

It is evident that the set A_0 is not empty, since contains random value $\xi = 1$. More interesting case is as A_0 contains more then one element.

Lemma 3.9 *On measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. If non negative random value ξ is such that $\sup_{P \in M} E^P \xi < \infty$, then $\{f_m = \text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a supermartingale relative to the convex set of equivalent measures M .*

Proof. From definition of $\text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}$ there exists a countable set D_m such that

$$\text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} = \sup_{P \in D_m} E^P \{\xi | \mathcal{F}_m\}, \quad m = \overline{0, \infty}.$$

The set $D = \bigcup_{m=0}^\infty D_m$ is also countable and

$$\text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} = \sup_{P \in D} E^P \{\xi | \mathcal{F}_m\}. \quad (3.25)$$

Really, since

$$\sup_{P \in D} E^P \{\xi | \mathcal{F}_m\} \geq \sup_{P \in D_m} E^P \{\xi | \mathcal{F}_m\} = \text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}. \quad (3.26)$$

From the other side,

$$\text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} \geq E^Q \{\xi | \mathcal{F}_m\}, \quad Q \in M. \quad (3.27)$$

The last gives

$$\text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\} \geq \sup_{P \in D} E^P \{\xi | \mathcal{F}_m\}. \quad (3.28)$$

The inequalities (3.26), (3.28) prove the needed. So, for all m we can choose the common set D . Let $D = \{\bar{P}_1, \dots, \bar{P}_n, \dots\}$. Due to Lemma 3.7, for every $Q \in \bar{M}_k$, where

$$\bar{M}_k = \{P \in M, P = \sum_{i=1}^k \alpha_i \bar{P}_i, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1\}, \quad (3.29)$$

we have

$$E^Q \left\{ \max_{1 \leq i \leq k} E^{\bar{P}_i} \{\xi | \mathcal{F}_n\} | \mathcal{F}_m \right\} \leq \max_{1 \leq i \leq k} E^{\bar{P}_i} \{\xi | \mathcal{F}_m\}, \quad n > m, \quad Q \in \bar{M}_k, \quad (3.30)$$

It is evident that $\max_{1 \leq i \leq k} E^{\bar{P}_i} \{\xi | \mathcal{F}_n\}$ tends to $\sup_{P \in D} E^P \{\xi | \mathcal{F}_n\}$ monotonously increasing, as $k \rightarrow \infty$. Fixing $Q \in \bar{M}_k \subset \bar{M}_{k+1}$ and tending k to the infinity in inequalities (3.30) we obtain

$$E^Q \left\{ \sup_{P \in D} E^P \{\xi | \mathcal{F}_n\} | \mathcal{F}_m \right\} \leq \sup_{P \in D} E^P \{\xi | \mathcal{F}_m\}, \quad n > m, \quad Q \in \bar{M}_k, \quad (3.31)$$

The last inequality implies that for every measure Q , belonging to the convex span, constructed on the set D , $\{f_m = \text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a supermartingale relative to the convex set of equivalent measures, generated by set D . Now, if a measure Q_0 does not belong to the convex span, constructed on the set D , then we can add it to the set D and repeat the proof made above. As a result, we proved that $\{f_m = \text{ess sup}_{P \in M} E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is also a supermartingale relative to the measure Q_0 . The Zorn Lemma [17] complete the proof of the Lemma 3.9.

Theorem 3.2 *On measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. For a random value $\xi \in A_0$ the random process $\{E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$, $P \in M$, is a local regular martingale relative to a convex set of equivalent measures M .*

Proof. Let P_1, \dots, P_n be a certain subset of measures from M . Denote by M_n a convex set of equivalent measures

$$M_n = \{P \in M, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i \geq 0, i = \overline{1, n}, \sum_{i=1}^n \alpha_i = 1\}. \quad (3.32)$$

Due to Lemma 3.8, $\{\bar{M}_m, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to the set of measures M_n , where $\bar{M}_m = \text{ess sup}_{P \in M_n} E^P \{\xi | \mathcal{F}_m\}$, $\xi \in A_0$. Let us consider an arbitrary measure $P_0 \in M$ and let

$$M_n^{P_0} = \{P \in M, P = \sum_{i=0}^n \alpha_i P_i, \alpha_i \geq 0, i = \overline{0, n}, \sum_{i=0}^n \alpha_i = 1\}. \quad (3.33)$$

Then $\{\bar{M}_m^{P_0}, \mathcal{F}_m\}_{m=0}^\infty$, where $\bar{M}_m^{P_0} = \text{ess sup}_{P \in M_n^{P_0}} E^P \{\xi | \mathcal{F}_m\}$, is a martingale relative to the set of measures $M_n^{P_0}$. It is evident that

$$\bar{M}_m \leq \bar{M}_m^{P_0}, \quad m = \overline{0, \infty}. \quad (3.34)$$

Since $E^P \bar{M}_m = E^P \bar{M}_m^{P_0} = 1$, $m = \overline{0, \infty}$, $P \in M_n$, the inequalities (3.34) give $\bar{M}_m = \bar{M}_m^{P_0}$. Analogously, $E^{P_0} \{\xi | \mathcal{F}_m\} \leq \bar{M}_m^{P_0}$. From equalities $E^{P_0} E^{P_0} \{\xi | \mathcal{F}_m\} = E^{P_0} \bar{M}_m^{P_0} = 1$ we obtain $E^{P_0} \{\xi | \mathcal{F}_m\} = \bar{M}_m^{P_0} = \bar{M}_m$. Since the measure P_0 is arbitrary it implies that $\{E^P \{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to all measures from M . Due to Theorem 2.7, it is a local regular supermartingale with random process $\bar{g}_m^0 = 0, m = \overline{0, \infty}$. The Theorem 3.2 is proved.

Theorem 3.3 *On measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. If $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is an adapted random process satisfying conditions*

$$f_m \leq f_{m-1}, \quad E^P \xi | f_m| < \infty, \quad P \in M \quad m = \overline{1, \infty}, \quad \xi \in A_0, \quad (3.35)$$

then the random process

$$\{f_m E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty, \quad P \in M, \quad (3.36)$$

is a local regular supermartingale relative to a convex set of equivalent measures M .

Proof. Due to Theorem 3.2, the random process $\{E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to the convex set of equivalent measures M . Therefore,

$$\begin{aligned} f_{m-1} E^P\{\xi|\mathcal{F}_{m-1}\} - E^P\{f_m E^P\{\xi|\mathcal{F}_m\}|\mathcal{F}_{m-1}\} = \\ E^P\{(f_{m-1} - f_m) E^P\{\xi|\mathcal{F}_m\}|\mathcal{F}_{m-1}\}, \quad m = \overline{1, \infty}. \end{aligned} \quad (3.37)$$

So, if to put $\bar{g}_m^0 = (f_{m-1} - f_m) E^P\{\xi|\mathcal{F}_m\}$, $m = \overline{1, \infty}$, then $\bar{g}_m^0 \geq 0$, it is \mathcal{F}_m -measurable and $E^P \bar{g}_m^0 \leq E^P \xi(|f_{m-1}| + |f_m|) < \infty$. It proves the needed statement.

Corollary 3.1 *If $f_m = \alpha$, $m = \overline{1, \infty}$, $\alpha \in R^1$, $\xi \in A_0$, then $\{\alpha E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular martingale. Assume that $\xi = 1$, then $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular supermartingale relative to a convex set of equivalent measures M .*

Denote by F_0 the set of adapted processes

$$F_0 = \{f = \{f_m\}_{m=0}^\infty, P(|f_m| < \infty) = 1, P \in M, f_m \leq f_{m-1}, m = \overline{1, \infty}\}.$$

For every $\xi \in A_0$ let us introduce the set of adapted processes

$$L_\xi =$$

$$\{\bar{f} = \{f_m E^P\{\xi|\mathcal{F}_m\}\}_{m=0}^\infty, \{f_m\}_{m=0}^\infty \in F_0, E^P \xi |f_m| < \infty, P \in M, m = \overline{1, \infty}\},$$

and

$$V = \bigcup_{\xi \in A_0} L_\xi.$$

Corollary 3.2 *Every random process from the set K , where*

$$K = \left\{ \sum_{i=1}^m C_i \bar{f}_i, \bar{f}_i \in V, C_i \geq 0, i = \overline{1, m}, m = \overline{1, \infty} \right\}, \quad (3.38)$$

is a local regular supermartingale relative to the convex set of equivalent measures M on a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it.

Proof. The proof is evident.

Theorem 3.4 *On measurable space $\{\Omega, \mathcal{F}\}$ and a filtration \mathcal{F}_n on it, let M be an arbitrary convex set of equivalent measures. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a non-negative uniformly integrable supermartingale relative to a convex set of equivalent measures M , then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set K .*

Proof. Necessity. It is evident that if $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ belongs to K , then it is a local regular supermartingale.

Sufficiency. Suppose that $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ is a local regular supermartingale. Then there exists nonnegative adapted process $\{\bar{g}_m^0\}_{m=1}^\infty$, $E^P \bar{g}_m^0 < \infty$, $m = \overline{1, \infty}$, and a martingale $\{M_m\}_{m=0}^\infty$, such that

$$f_m = M_m - \sum_{i=1}^m \bar{g}_i^0, \quad m = \overline{0, \infty}.$$

Then $M_m \geq 0$, $m = \overline{0, \infty}$, $E^P M_m < \infty$, $P \in M$. Since $0 < E^P M_m = f_0 < \infty$ we have $E^P \sum_{i=1}^m \bar{g}_i^0 < f_0$. Let us put $g_\infty = \lim_{m \rightarrow \infty} \sum_{i=1}^m \bar{g}_i^0$. Using uniform integrability of f_m , we can pass to the limit in the equality

$$E^P(f_m + \sum_{i=1}^m \bar{g}_i^0) = f_0, \quad P \in M,$$

as $m \rightarrow \infty$. Passing to the limit in the last equality, as $m \rightarrow \infty$, we obtain

$$E^P(f_\infty + g_\infty) = f_0, \quad P \in M.$$

Introduce into consideration a random value $\xi = \frac{f_\infty + g_\infty}{f_0}$. Then $E^P \xi = 1$, $P \in M$. From here we obtain that $\xi \in A_0$ and

$$M_m = f_0 E^P\{\xi | \mathcal{F}_m\}, \quad m = \overline{0, \infty}.$$

Let us put $\bar{f}_m^2 = -\sum_{i=1}^m \bar{g}_i^0$. It is easy to see that an adapted random process $\bar{f}_2 = \{\bar{f}_m^2, \mathcal{F}_m\}_{m=0}^\infty$ belongs to F_0 . Therefore, for the supermartingale $f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty$ the representation

$$f = \bar{f}_1 + \bar{f}_2,$$

is valid, where $\bar{f}_1 = \{f_0 E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ belongs to L_ξ with $\xi = \frac{f_\infty + g_\infty}{f_0}$ and $\bar{f}_m^1 = f_0$, $m = \overline{0, \infty}$. The same is valid for \bar{f}_2 with $\xi = 1$. This implies that f belongs to the set K . The Theorem 3.4 is proved.

Corollary 3.3 *Let f_N , $N < \infty$, be a \mathcal{F}_N -measurable integrable random value, $\sup_{P \in M} E^P |f_N| < \infty$, and let there exist $\alpha_0 \in R^1$ such that*

$$-\alpha_0 M_N + f_N \leq 0, \quad \omega \in \Omega,$$

where $\{M_m, \mathcal{F}_m\}_{m=0}^\infty = \{E^P\{\xi | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$, $\xi \in A_0$. Then a supermartingale $\{f_m^0 + \bar{f}_m\}_{m=0}^\infty$ is local regular one relative to a convex set of equivalent measures M , where

$$\begin{aligned} f_m^0 &= \alpha_0 M_m, \\ \bar{f}_m &= \begin{cases} 0, & m < N, \\ f_N - \alpha_0 M_N, & m \geq N. \end{cases} \end{aligned}$$

Proof. It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0$, $m = \overline{0, \infty}$. Therefore, the supermartingale

$$f_m^0 + \bar{f}_m = \begin{cases} \alpha_0 M_m, & m < N, \\ f_N, & m = N, \\ f_N - \alpha_0 M_N + \alpha_0 M_m, & m > N, \end{cases}$$

is local regular one relative to a convex set of equivalent measures M . The Corollary 3.3 is proved.

4 Optional decomposition for non negative supermartingales.

In this section we introduce the notion of complete set of equivalent measures and prove that non negative supermartingales are local regular with respect to this set of measures. For this purpose we are needed the next auxiliary statement.

Theorem 4.1 *The necessary and sufficient condition of local regularity of non negative supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures M is the existence of \mathcal{F}_m -measurable random value $\xi_m^0 \in A_0$ such that*

$$\frac{f_m}{f_{m-1}} \leq \xi_m^0, \quad E^P\{\xi_m^0 | \mathcal{F}_{m-1}\} = 1, \quad P \in M, \quad m = \overline{1, \infty}. \quad (4.1)$$

Proof. The necessity. Let $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ be a local regular supermartingale. Then there exists non negative adapted random process $\{g_m\}_{m=0}^\infty$, $g_0 = 0$, such that $\sup_{P \in M} E^P g_m < \infty$,

$$f_{m-1} - E^P\{f_m | \mathcal{F}_{m-1}\} = E^P\{g_m | \mathcal{F}_{m-1}\}, \quad P \in M, \quad m = \overline{1, \infty}. \quad (4.2)$$

Let us put $\xi_m^0 = \frac{f_m + g_m}{f_{m-1}}$, $m = \overline{1, \infty}$. Then from (4.2) $E^P\{\xi_m^0 | \mathcal{F}_{m-1}\} = 1$, $P \in M$, $m = \overline{1, \infty}$. It is evident that inequalities (4.1) are valid.

The sufficiency. Suppose that conditions of the Theorem 4.1 are valid. Then $f_m \leq f_{m-1} + f_{m-1}(\xi_m^0 - 1)$. Introduce denotation $g_m = -f_m + f_{m-1}\xi_m^0$. Then $g_m \geq 0$, $\sup_{P \in M} E^P g_m \leq \sup_{P \in M} E^P f_m + \sup_{P \in M} E^P f_{m-1} < \infty$, $m = \overline{1, \infty}$. The last inequalities and equality give

$$f_m = f_0 + \sum_{i=1}^m f_{i-1}(\xi_i^0 - 1) - \sum_{i=1}^m g_i, \quad m = \overline{1, \infty}. \quad (4.3)$$

Let us consider $\{M_m, \mathcal{F}_m\}_{m=0}^\infty$, where $M_m = f_0 + \sum_{i=1}^m f_{i-1}(\xi_i^0 - 1)$. Then $E^P\{M_m | \mathcal{F}_{m-1}\} = M_{m-1}$, $P \in M$, $m = \overline{1, \infty}$. The Theorem 4.1 is proved.

4.1 Space of finite set of elementary events.

In this subsection we assume that a space of elementary events Ω is finite, that is, $N_0 = |\Omega| < \infty$, and we give new proof of optional decomposition for non negative supermartingale relative to some convex set of equivalent measures.

Let \mathcal{F} be some algebra of subsets of Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$. Denote by M a set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. Further, we assume that a set A_0 contains an element $\xi_0 \neq 1$. It is evident that every algebra \mathcal{F}_n is generated by sets A_i^n , $i = \overline{1, N_n}$, $A_i^n \cap A_j^n = \emptyset$, $i \neq j$, $N_n < \infty$, $\bigcup_{i=1}^{N_n} A_i^n = \Omega$, $n = \overline{1, N}$. Between the sets A_i^n and A_j^{n-1} the relations $A_j^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid, where $I_j \subseteq T_n$, $T_n = \{1, 2, \dots, N_n\}$, $I_s \cap$

$I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{N_{n-1}} I_j = T_n$. Let $m_n = E^P\{\xi_0 | \mathcal{F}_n\}$, $P \in M$, $n = \overline{1, N}$. Then for m_n

the representation

$$m_n = \sum_{i=1}^{N_n} m_i^n \chi_{A_i^n}(\omega), \quad n = \overline{1, N}, \quad (4.4)$$

is valid. Consider the difference $m_n - m_{n-1}$. Then

$$\begin{aligned} m_n - m_{n-1} &= \sum_{s=1}^{N_{n-1}} \sum_{j \in I_s} (m_j^n - m_s^{n-1}) \chi_{A_j^n}(\omega) = \\ &= \sum_{s=1}^{N_{n-1}} \sum_{j=1}^{N_n} \chi_{I_s}(j) (m_j^n - m_s^{n-1}) \chi_{A_j^n} = \sum_{j=1}^{N_n} [m_j^n - \sum_{s=1}^{N_{n-1}} \chi_{I_s}(j) m_s^{n-1}] \chi_{A_j^n}. \end{aligned} \quad (4.5)$$

Introduce the set of numbers $a_{js}^n = m_j^n - m_s^{n-1}$, $j \in I_s$, $s = \overline{1, N_{n-1}}$, and sets $I_s^- = \{j \in I_s, a_{js}^n \leq 0\}$, $I_s^+ = \{j \in I_s, a_{js}^n > 0\}$, $I^- = \bigcup_{s=1}^{N_{n-1}} I_s^-$, $I^+ = \bigcup_{s=1}^{N_{n-1}} I_s^+$. Then

$$m_n - m_{n-1} = \sum_{j \in I^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I^+} d_j^n \chi_{A_j^n}(\omega), \quad (4.6)$$

$$\sum_{j \in I^-} \chi_{A_j^n}(\omega) + \sum_{j \in I^+} \chi_{A_j^n}(\omega) = 1, \quad (4.7)$$

where $d_j^n = a_{js}^n$, as $j \in I_s^-$, or $j \in I_s^+$. From equalities (4.6), (4.7) we obtain

$$\sum_{j \in I^-} d_j^n P(A_j^n) + \sum_{j \in I^+} d_j^n P(A_j^n) = 0, \quad P \in M, \quad (4.8)$$

$$\sum_{j \in I^-} P(A_j^n) + \sum_{j \in I^+} P(A_j^n) = 1, \quad \in M. \quad (4.9)$$

Denote by M_n the contraction of the set of measures M on the algebra \mathcal{F}_n . Introduce into the set M_n metrics

$$\rho_n(P_1, P_2) = \sum_{j \in I^-} |P_1(A_j^n) - P_2(A_j^n)| + \quad (4.10)$$

$$\sum_{j \in I^+} |P_1(A_j^n) - P_2(A_j^n)|, \quad n = \overline{1, N}.$$

Definition 4.1 On a measurable space $\{\Omega, \mathcal{F}\}$, a set of measure M we call complete if for every $1 \leq n \leq N$ the closure of the set of measures M_n in metrics (4.10) contains measures

$$P_{ij}^n(A) = \begin{cases} 0, & A \neq A_i^n, A_j^n, \\ \frac{d_j^n}{-d_i^n + d_j^n}, & A = A_i^n, \\ \frac{-d_i^n}{-d_i^n + d_j^n}, & A = A_j^n, \end{cases} \quad (4.11)$$

for every $i \in I^-$ and $j \in I^+$.

Lemma 4.1 *Let a family of measures M be complete and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non negative \mathcal{F}_n -measurable random value $\xi_n = \sum_{i=1}^{N_n} C_i^n \chi_{A_i^n}$ there exists a real number α_n such that*

$$\frac{\sum_{i=1}^{N_n} C_i^n \chi_{A_i^n}}{\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n)} \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = \overline{1, N}. \quad (4.12)$$

Proof. On the set \bar{M}_n , a functional $\varphi(P) = \sum_{i=1}^{N_n} C_i^n P(A_i^n)$ is continuous one, where \bar{M}_n is the closure of the set M_n in the metrics $\rho_n(P_1, P_2)$. From this it follows that the equality

$$\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) = \sup_{P \in \bar{M}_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) \quad (4.13)$$

is valid. Denote by $f_i^n = \frac{C_i^n}{\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n)}$, $i = \overline{1, N_n}$. Then

$$\sum_{i=1}^{N_n} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n. \quad (4.14)$$

In every set I_s^- there are strictly negative elements and in the every set I_s^+ there are strictly positive elements. For those $i \in I^-$ for which $d_i^n < 0$ and those $j \in I^+$ for which $d_j^n > 0$ the inequality (4.14) is as follows

$$f_i^n \frac{d_j^n}{-d_i^n + d_j^n} + \frac{-d_i^n}{-d_i^n + d_j^n} f_j^n \leq 1, \quad (4.15)$$

$$d_i^n < 0, \quad i \in I^-, \quad d_j^n > 0, \quad j \in I^+.$$

From (4.15) we obtain inequalities

$$f_j^n \leq 1 + \frac{1 - f_i^n}{-d_i^n} d_j^n, \quad d_i^n < 0, \quad i \in I^-, \quad d_j^n > 0, \quad j \in I^+. \quad (4.16)$$

Since the inequalities (4.16) are valid for every $\frac{1 - f_i^n}{-d_i^n}$, as $d_i^n < 0$, and since the set of such elements is finite, then if to denote

$$\alpha_n = \min_{\{i, d_i^n < 0\}} \frac{1 - f_i^n}{-d_i^n},$$

then we have

$$f_j^n \leq 1 + \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I^+. \quad (4.17)$$

From the definition of α_n we obtain inequalities

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I^-.$$

Now if $d_i^n = 0$ for some $i \in I^-$, then in this case $f_i^n \leq 1$. All these inequalities give

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad i \in I^- \cup I^+. \quad (4.18)$$

Multiplying on $\chi_{A_i^n}$ the inequalities (4.18) and summing over all $i \in I^- \cup I^+$ we obtain the needed inequality. The Lemma 4.1 is proved.

Theorem 4.2 *Suppose that conditions of the Lemma 4.1 are valid. Then for every non negative supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ optional decomposition is valid.*

Proof. Consider random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 4.1

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n(m_n - m_{n-1}) = \xi_n^0, \quad n = \overline{1, N}.$$

It is evident that $E^P\{\xi_n^0 | \mathcal{F}_{n-1}\} = 1$, $P \in M$, $n = \overline{1, N}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, \quad n = \overline{1, N}. \quad (4.19)$$

The Theorem 4.1 and inequalities (4.19) prove the Lemma 4.2.

4.2 Countable set of elementary events.

In this subsection we generalize the results of the previous subsection onto the countable space of elementary events.

Let \mathcal{F} be some σ -algebra of subsets of the countable set of elementary events Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be a certain increasing set of σ -algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote by M a set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. Further, we assume that the set A_0 contains an element $\xi_0 \neq 1$. Suppose that σ -algebra \mathcal{F}_n is generated by sets A_i^n , $i = \overline{1, \infty}$, $A_i^n \cap A_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} A_i^n = \Omega$, $n = \overline{1, \infty}$. We also assume that between the sets A_i^n and A_j^{n-1} the relations $A_j^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid,

where $I_j \subseteq N_0 = \{1, 2, \dots, n, \dots\}$, $I_s \cap I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{\infty} I_j = N_0$. Introduce into consideration a martingale $m_n = E^P\{\xi_0 | \mathcal{F}_n\}$, $P \in M$, $n = \overline{1, \infty}$. Then for m_n the representation

$$m_n = \sum_{i=1}^{\infty} m_i^n \chi_{A_i^n}(\omega), \quad n = \overline{1, \infty}, \quad (4.20)$$

is valid. Consider the difference $m_n - m_{n-1}$. Then

$$m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{j \in I_s} (m_j^n - m_s^{n-1}) \chi_{A_j^n}(\omega) =$$

$$\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \chi_{I_s}(j)(m_j^n - m_s^{n-1}) \chi_{A_j^n} = \sum_{j=1}^{\infty} [m_j^n - \sum_{s=1}^{\infty} \chi_{I_s}(j) m_s^{n-1}] \chi_{A_j^n}. \quad (4.21)$$

Introduce the set of numbers $a_{js}^n = m_j^n - m_s^{n-1}$, $j \in I_s$, $s = \overline{1, \infty}$, and sets $I_s^- = \{j \in I_s, a_{js}^n \leq 0\}$, $I_s^+ = \{j \in I_s, a_{js}^n > 0\}$, $I^- = \bigcup_{s=1}^{\infty} I_s^-$, $I^+ = \bigcup_{s=1}^{\infty} I_s^+$. Then

$$m_n - m_{n-1} = \sum_{j \in I^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I^+} d_j^n \chi_{A_j^n}(\omega), \quad (4.22)$$

$$\sum_{j \in I^-} \chi_{A_j^n}(\omega) + \sum_{j \in I^+} \chi_{A_j^n}(\omega) = 1, \quad (4.23)$$

where $d_j^n = a_{js}^n$, as $j \in I_s^-$, or $j \in I_s^+$. From equalities (4.22), (4.23) we obtain

$$\sum_{j \in I^-} d_j^n P(A_j^n) + \sum_{j \in I^+} d_j^n P(A_j^n) = 0, \quad P \in M, \quad (4.24)$$

$$\sum_{j \in I^-} P(A_j^n) + \sum_{j \in I^+} P(A_j^n) = 1, \quad P \in M. \quad (4.25)$$

Denote by M_n the contraction of the set of measures M on the σ -algebra \mathcal{F}_n . Introduce into the set M_n metrics

$$\rho_n(P_1, P_2) = \sum_{j \in I^-} |P_1(A_j^n) - P_2(A_j^n)| + \sum_{j \in I^+} |P_1(A_j^n) - P_2(A_j^n)|, \quad (4.26)$$

$$n = \overline{1, \infty}.$$

Definition 4.2 On a measurable space $\{\Omega, \mathcal{F}\}$, a set of measure M we call complete if for every $1 \leq n < \infty$ the closure of the set of measures M_n in metrics (4.26) contains measures

$$P_{ij}^n(A) = \begin{cases} 0, & A \neq A_i^n, A_j^n, \\ \frac{d_j^n}{-d_i^n + d_j^n}, & A = A_i^n, \\ \frac{-d_i^n}{-d_i^n + d_j^n}, & A = A_j^n, \end{cases} \quad (4.27)$$

for every $i \in I^-$ and $j \in I^+$.

Lemma 4.2 Let a family of measures M be complete and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non negative bounded \mathcal{F}_n -measurable random value

$\xi_n = \sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}$ there exists real number α_n such that

$$\frac{\sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)} \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = \overline{1, \infty}. \quad (4.28)$$

Proof. On the set \bar{M}_n , a functional $\varphi(P) = \sum_{i=1}^{\infty} C_i^n P(A_i^n)$ is continuous one, where \bar{M}_n is the closure of the set M_n in metrics $\rho_n(P_1, P_2)$. From this it follows that the equality

$$\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n) = \sup_{P \in \bar{M}_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n) \quad (4.29)$$

is valid. Denote by $f_i^n = \frac{C_i^n}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)}$, $i = \overline{1, \infty}$. Then

$$\sum_{i=1}^{\infty} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n.$$

The last inequalities can be written in the form

$$\sum_{i \in I^-} f_i^n P(A_i^n) + \sum_{i \in I^+} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n. \quad (4.30)$$

In every set I_s^- there are strictly negative elements and in the every set I_s^+ there are strictly positive elements. For those $i \in I^-$ for which $d_i^n < 0$ and those $j \in I^+$ for which $d_j^n > 0$ the inequality (4.30) is as follows

$$f_i^n \frac{d_j^n}{-d_i^n + d_j^n} + \frac{-d_i^n}{-d_i^n + d_j^n} f_j^n \leq 1, \quad (4.31)$$

$$d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+.$$

From (4.31) we obtain inequalities

$$f_j^n \leq 1 + \frac{1 - f_i^n}{-d_i^n} d_j^n, \quad d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+. \quad (4.32)$$

Two cases are possible: a) for all $i \in I^-$, $f_i^n \leq 1$; b) there exists $i \in I^-$ such that $f_i^n > 1$. First, let us consider the case a).

Since inequalities (4.32) are valid for every $\frac{1-f_i^n}{-d_i^n}$, as $d_i^n < 0$, and $f_i^n \leq 1, i \in I^-$, then if to denote

$$\alpha_n = \inf_{\{i, d_i^n < 0\}} \frac{1 - f_i^n}{-d_i^n},$$

we have $0 \leq \alpha_n < \infty$ and

$$f_j^n \leq 1 + \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I^+. \quad (4.33)$$

From the definition of α_n we obtain inequalities

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I^-.$$

Now, if $d_i^n = 0$ for some $i \in I^-$, then in this case $f_i^n \leq 1$. All these inequalities give

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad i \in I^- \cup I^+. \quad (4.34)$$

Consider the case b). From the inequality (4.32) we obtain

$$f_j^n \leq 1 - \frac{1 - f_i^n}{d_i^n} d_j^n, \quad d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+. \quad (4.35)$$

The last inequalities give

$$\frac{1 - f_i^n}{d_i^n} \leq \min_{\{j, d_j^n > 0\}} \frac{1}{d_j^n} < \infty, \quad d_i^n < 0, \quad i \in I^-. \quad (4.36)$$

Let us define $\alpha_n = \sup_{\{i, d_i^n < 0\}} \frac{1 - f_i^n}{d_i^n} < \infty$. Then from (4.35) we obtain

$$f_j^n \leq 1 - \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I^+. \quad (4.37)$$

From the definition of α_n we have

$$f_i^n \leq 1 - \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I^-. \quad (4.38)$$

The inequalities (4.37), (4.38) give

$$f_j^n \leq 1 - \alpha_n d_j^n, \quad j \in I^- \cup I^+. \quad (4.39)$$

Multiplying on $\chi_{A_j^n}$ the inequalities (4.39) and summing over all $j \in I^- \cup I^+$ we obtain the needed inequality. The Lemma 4.2 is proved.

Theorem 4.3 *Suppose that conditions of the Lemma 4.2 are valid. Then for every non negative supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$, satisfying conditions*

$$\sup_{P \in M} E^P f_m < \infty, \quad \frac{f_m}{f_{m-1}} \leq C_m < \infty, \quad m = \overline{1, \infty}, \quad (4.40)$$

optional decomposition is valid.

Proof. Consider random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 4.2

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_n^0.$$

It is evident that $E^P \{\xi_n^0 | \mathcal{F}_{n-1}\} = 1$, $P \in M$, $n = \overline{1, \infty}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, \quad n = \overline{1, N}. \quad (4.41)$$

The Theorem 4.1 and inequalities (4.41) prove the Lemma 4.3.

4.3 An arbitrary space of elementary events.

In this subsection we consider an arbitrary space of elementary events and prove optional decomposition for non negative supermartingales.

Let \mathcal{F} be some σ -algebra of subsets of the set of elementary events Ω and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of σ -algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote by M a set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. We assume that σ -algebras \mathcal{F}_n , $n = \overline{1, \infty}$, and \mathcal{F} are complete relative to all measure $P \in M$. Further, we suppose that a set A_0 contains an element $\xi_0 \neq 1$. Let $m_n = E^P\{\xi_0 | \mathcal{F}_n\}$, $P \in M$, $n = \overline{1, \infty}$. Then for m_n the representation

$$m_n = \sum_{i=1}^{\infty} m_i^n \chi_{A_i^n}(\omega), \quad n = \overline{1, \infty}, \quad (4.42)$$

is valid for some $A_i^n \in \mathcal{F}_n$, A_i^n , $i = \overline{1, \infty}$, $A_i^n \cap A_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} A_i^n = \Omega$, $n = \overline{1, \infty}$.

Really, let us consider a sequence of random values $m_n = E^P\{\xi_0 | \mathcal{F}_n\}$, $P \in M$, $n = \overline{1, \infty}$. It is evident that $E^P\{m_n | \mathcal{F}_{n-1}\} = m_{n-1}$. For every random value m_n there exists not more then a countable set of non negative real number $m_s^n \geq 0$ such that $P(A_s^n) > 0$, where $A_s^n = \{\omega \in \Omega, m_n = m_s^n\}$. It is evident that $A_i^n \cap A_j^n = \emptyset$, $i \neq j$. Since m_n is defined on all Ω , then $P(\bigcup_{i=1}^{\infty} A_i^n) = 1$. From this it follows

that $P(\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n) = 0$. The set $\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n$ we can join, for example, to A_1^n and

to put $m_n = m_1^n$, $\omega \in A_1^n \cup (\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n)$. If to change denotation we come to the

above statement. Further, let us prove that we can choose the sets A_i^n such that the relations $A_j^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid, where $I_j \subseteq N_0 = \{1, 2, \dots, n, \dots\}$, $I_s \cap I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{\infty} I_j = N_0$. Really, if it is not so then we can choose countable set of

subsets $B_{ij}^n = A_i^n \cap A_j^{n-1}$, $i, j = \overline{1, \infty}$. It is evident that $\bigcup_{j=1}^{\infty} B_{ij}^n = A_i^n$, $\bigcup_{i=1}^{\infty} B_{ij}^n = A_j^{n-1}$.

For fixed j denote by I_j those indexes i for which $P(B_{ij}^n) > 0$. Then $\bigcup_{i \in I_j} B_{ij}^n = A_j^{n-1}$.

Let us define on B_{ij}^n random value putting $m_{ij}^n = m_i^n$, $\omega \in B_{ij}^n$, $i \in I_j$, $j = \overline{1, \infty}$. Then

$$\sum_{j=1}^{\infty} \sum_{i \in I_j} m_{ij}^n \chi_{B_{ij}^n}(\omega) = \sum_{j=1}^{\infty} \sum_{i \in I_j} m_i^n \chi_{B_{ij}^n}(\omega) = m_n, \quad n = \overline{1, \infty}. \quad (4.43)$$

Taking into account these facts, further without loss of generality we suppose that between the sets A_i^n and A_j^{n-1} the relations $A_j^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid, where

$I_j \subseteq N_0 = \{1, 2, \dots, n, \dots\}$, $I_s \cap I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{\infty} I_j = N_0$.

Consider the difference $m_n - m_{n-1}$. Then

$$m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{j \in I_s} (m_j^n - m_s^{n-1}) \chi_{A_j^n}(\omega) =$$

$$\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \chi_{I_s}(j) (m_j^n - m_s^{n-1}) \chi_{A_j^n} = \sum_{j=1}^{\infty} [m_j^n - \sum_{s=1}^{\infty} \chi_{I_s}(j) m_s^{n-1}] \chi_{A_j^n}. \quad (4.44)$$

Introduce the set of numbers $a_{js}^n = m_j^n - m_s^{n-1}$, $j \in I_s$, $s = \overline{1, \infty}$, and sets $I_s^- = \{j \in I_s, a_{js}^n \leq 0\}$, $I_s^+ = \{j \in I_s, a_{js}^n > 0\}$, $I^- = \bigcup_{s=1}^{\infty} I_s^-$, $I^+ = \bigcup_{s=1}^{\infty} I_s^+$. Then

$$m_n - m_{n-1} = \sum_{j \in I^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I^+} d_j^n \chi_{A_j^n}(\omega), \quad (4.45)$$

$$\sum_{j \in I^-} \chi_{A_j^n}(\omega) + \sum_{j \in I^+} \chi_{A_j^n}(\omega) = 1, \quad (4.46)$$

where $d_j^n = a_{js}^n$, as $j \in I_s^-$, or $j \in I_s^+$.

Let a countable set of subsets $D_j^n \in \mathcal{F}_n$, $j = \overline{1, \infty}$, be such that $D_i^n \cap D_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} D_i^n = \Omega$, $n = \overline{1, \infty}$. Denote by $\tilde{\mathcal{F}}_n^D \subseteq \mathcal{F}_n$ a sub σ -algebra of the σ -algebra \mathcal{F}_n , generated by the countable set of subsets $D_j^n \in \mathcal{F}_n$, $j = \overline{1, \infty}$.

Let M_n^D be the contraction of the set of measures M on the sub σ -algebra $\tilde{\mathcal{F}}_n^D \subseteq \mathcal{F}_n$. Introduce into the set M_n^D metrics

$$\rho_n^D(P_1, P_2) = \sum_{j \in I^-} |P_1(D_j^n) - P_2(D_j^n)| + \sum_{j \in I^+} |P_1(D_j^n) - P_2(D_j^n)|, \quad (4.47)$$

$$n = \overline{1, \infty}.$$

Definition 4.3 On a measurable space $\{\Omega, \mathcal{F}\}$, a set of measure M we call complete if for every $1 \leq n < \infty$ and every countable set of subsets $D_j^n \in \mathcal{F}_n$, $j = \overline{1, \infty}$, $D_i^n \cap D_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} D_i^n = \Omega$, $n = \overline{1, \infty}$, the closure in metrics (4.47) of the set of measures M_n^D contains measures

$$P_{ij}^n(B) = \begin{cases} 0, & B \neq D_i^n, D_j^n, \\ \frac{d_j^n}{-d_i^n + d_j^n}, & B = D_i^n, \\ \frac{-d_i^n}{-d_i^n + d_j^n}, & B = D_j^n, \end{cases} \quad (4.48)$$

for every $i \in I^-$ and $j \in I^+$.

Lemma 4.3 Let a family of measures M be complete and the set A_0 contains an element $\xi_0 \neq 1$. Then for every non negative bounded \mathcal{F}_n -measurable random value ξ_n there exists a real number α_n such that

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n (m_n - m_{n-1}), \quad n = \overline{1, \infty}. \quad (4.49)$$

Proof. For the random value ξ_n the representation $\sum_{j=1}^{\infty} \xi_j^n \chi_{V_j^n}$ is valid, where $V_j^n \in \mathcal{F}_n$, $j = \overline{1, \infty}$, $V_i^n \cap V_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} V_i^n = \Omega$, $n = \overline{1, \infty}$. Introduce into consideration the countable set of subsets $U_{ij}^n = A_i^n \cap V_j^n$, $i, j = \overline{1, \infty}$. It is evident that $\bigcup_{i,j=1}^{\infty} U_{ij}^n = \Omega$, $U_{ij}^n \cap U_{rs}^n = \emptyset$, $\{ij\} \neq \{rs\}$.

Then

$$m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{i \in I^-} d_i^n \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_j^n \chi_{A_j^n \cap V_t^n}(\omega), \quad (4.50)$$

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} \chi_{A_j^n \cap V_t^n}(\omega) = 1, \quad (4.51)$$

where $d_j^n = a_{js}^n$, as $j \in I_s^-$, or $j \in I_s^+$. From equalities (4.50), (4.51) we obtain

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} d_i^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_j^n P(A_j^n \cap V_t^n) = 0, \quad P \in M, \quad (4.52)$$

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} P(A_j^n \cap V_t^n) = 1, \quad P \in M. \quad (4.53)$$

The random value ξ_n can be written in the form

$$\xi_n = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_s^n \chi_{A_j^n \cap V_s^n}. \quad (4.54)$$

Let M_n^U be the contraction of the set of measures M on the sub σ -algebra $\tilde{\mathcal{F}}_n^U$, generated by the countable set of subsets U_{ij}^n , $i, j = \overline{1, \infty}$. On the set \bar{M}_n^U , a functional $\varphi(P) = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_s^n P(A_j^n \cap V_s^n)$, $P \in \bar{M}_n^U$, is continuous one in the metrics $\rho_n^U(P_1, P_2)$, where \bar{M}_n^U is the closure of the set M_n^U in the metrics. From this it follows that the equality

$$\sup_{P \in M_n^U} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_s^n P(A_j^n \cap V_s^n) = \sup_{P \in \bar{M}_n^U} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_s^n P(A_j^n \cap V_s^n) \quad (4.55)$$

is valid.

Denote by $f_s^n = \frac{\xi_s^n}{\sup_{P \in M_n} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_s^n P(A_j^n \cap V_s^n)}$, $s = \overline{1, \infty}$. Then

$$\sum_{j=1}^{\infty} \sum_{s=1}^{\infty} f_s^n P(A_j^n \cap V_s^n) \leq 1, \quad P \in \bar{M}_n^U. \quad (4.56)$$

The last inequalities can be written in the form

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} f_s^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} f_t^n P(A_j^n \cap V_t^n) \leq 1, \quad P \in \bar{M}_n^U. \quad (4.57)$$

Let us write equalities (4.50), (4.51) in more general form

$$m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{i \in I^-} d_{is}^n \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_{jt}^n \chi_{A_j^n \cap V_t^n}(\omega), \quad (4.58)$$

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} \chi_{A_j^n \cap V_t^n}(\omega) = 1, \quad (4.59)$$

where $d_{is}^n = d_i^n$, $s = \overline{1, \infty}$, $d_{jt}^n = d_j^n$, $t = \overline{1, \infty}$.

From equalities (4.58), (4.59) we obtain

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} d_{is}^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_{jt}^n P(A_j^n \cap V_t^n) = 0, \quad P \in M, \quad (4.60)$$

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} P(A_j^n \cap V_t^n) = 1, \quad P \in M. \quad (4.61)$$

Let us write inequality (4.57) in the form

$$\sum_{s=1}^{\infty} \sum_{i \in I^-} f_{is}^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} f_{jt}^n P(A_j^n \cap V_t^n) \leq 1, \quad P \in \bar{M}_n^U, \quad (4.62)$$

where $f_{is}^n = f_s^n$, $i = \overline{1, \infty}$, $f_{jt}^n = f_t^n$, $j = \overline{1, \infty}$.

Due to completeness of the set of measures M , for those $i \in I^-$ for which $d_{is}^n < 0$, $s = \overline{1, \infty}$, and those $j \in I^+$ for which $d_{jt}^n > 0$, $t = \overline{1, \infty}$, the inequality (4.62) is as follows

$$f_{is}^n \frac{d_{jt}^n}{-d_{is}^n + d_{jt}^n} + \frac{-d_{is}^n}{-d_{is}^n + d_{jt}^n} f_{jt}^n \leq 1, \quad d_{is}^n < 0, \quad s = \overline{1, \infty}, \quad (4.63)$$

$$d_{jt}^n > 0, \quad t = \overline{1, \infty}.$$

From inequalities (4.63) we obtain

$$f_{jt}^n \leq 1 + \frac{1 - f_{is}^n}{-d_{is}^n} d_{jt}^n, \quad d_{is}^n < 0, \quad s = \overline{1, \infty}, \quad d_{jt}^n > 0, \quad t = \overline{1, \infty}. \quad (4.64)$$

Two cases are possible: a) $f_{is}^n \leq 1$, $i \in I^-$, $s = \overline{1, \infty}$; b) there exists $i \in I^-$ such that $f_{is}^n > 1$, $s = \overline{1, \infty}$. Consider the case a).

Since inequalities (4.64) are valid for every $\frac{1-f_{is}^n}{-d_{is}^n}$, as $d_{is}^n < 0$, $s = \overline{1, \infty}$, and $f_{is}^n \leq 1$, $i \in I^-$, $s = \overline{1, \infty}$, then if to denote

$$\alpha_n = \inf_{\{is, d_{is}^n < 0\}} \frac{1 - f_{is}^n}{-d_{is}^n},$$

we have $0 \leq \alpha_n < \infty$, and

$$f_{jt}^n \leq 1 + \alpha_n d_{jt}^n, \quad d_{jt}^n > 0, \quad j \in I^+, \quad t = \overline{1, \infty}. \quad (4.65)$$

From the definition of α_n we obtain inequalities

$$f_{is}^n \leq 1 + \alpha_n d_{is}^n, \quad d_{is}^n < 0, \quad i \in I^-, \quad s = \overline{1, \infty}.$$

Now if $d_{is}^n = 0$ for some $i \in I^-$, $s = \overline{1, \infty}$, then in this case $f_{is}^n \leq 1$. All these inequalities give

$$f_{is}^n \leq 1 + \alpha_n d_{is}^n, \quad i \in I^- \cup I^+, \quad s = \overline{1, \infty}. \quad (4.66)$$

Consider the case b). From the inequality (4.64) we obtain

$$f_{jt}^n \leq 1 - \frac{1 - f_{is}^n}{d_{is}^n} d_{jt}^n, \quad d_{is}^n < 0, \quad i \in I^-, \quad s = \overline{1, \infty}, \quad (4.67)$$

$$d_{jt}^n > 0, \quad j \in I^+, \quad t = \overline{1, \infty}.$$

The last inequality gives

$$\frac{1 - f_{is}^n}{d_{is}^n} \leq \min_{\{jt, d_{jt}^n > 0\}} \frac{1}{d_{jt}^n} < \infty, \quad d_{is}^n < 0, \quad i \in I^-, \quad s = \overline{1, \infty}. \quad (4.68)$$

Let us define $\alpha_n = \sup_{\{is, d_{is}^n < 0\}} \frac{1 - f_{is}^n}{d_{is}^n} < \infty$. Then from (4.67) we obtain

$$f_{jt}^n \leq 1 - \alpha_n d_{jt}^n, \quad d_{jt}^n > 0, \quad j \in I^+, \quad t = \overline{1, \infty}. \quad (4.69)$$

From the definition of α_n we obtain

$$f_{is}^n \leq 1 - \alpha_n d_{is}^n, \quad d_{is}^n < 0, \quad i \in I^-, \quad s = \overline{1, \infty}. \quad (4.70)$$

The inequalities (4.69), (4.70) give

$$f_{is}^n \leq 1 - \alpha_n d_{is}^n, \quad i \in I^- \cup I^+, \quad s = \overline{1, \infty}. \quad (4.71)$$

Multiplying on $\chi_{A_i^n \cap V_s^n}$ the inequalities (4.71) and summing over all $i \in I^- \cup I^+$ and $s = \overline{1, \infty}$ we obtain

$$\begin{aligned} \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} f_{is} \chi_{A_i^n \cap V_s^n} &= \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} f_s \chi_{A_i^n \cap V_s^n} = \frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq \\ 1 - \alpha_n \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} d_{is} \chi_{A_i^n \cap V_s^n} &= 1 - \alpha_n (m_n - m_{n-1}). \end{aligned} \quad (4.72)$$

The Lemma 4.3 is proved.

Theorem 4.4 Suppose that conditions of the Lemma 4.3 are valid. Then for every non negative supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$, satisfying conditions

$$\sup_{P \in M} E^P f_m < \infty, \quad \frac{f_m}{f_{m-1}} \leq C_m < \infty, \quad m = \overline{1, \infty}, \quad (4.73)$$

optional decomposition is valid.

Proof. Consider random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 4.3

$$\frac{\xi_n}{\sup_{P \in M} E^P \xi_n} \leq 1 + \alpha_n(m_n - m_{n-1}) = \xi_n^0.$$

It is evident that $E^P\{\xi_n^0 | \mathcal{F}_{n-1}\} = 1$, $P \in M$, $n = \overline{1, \infty}$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then

$$\frac{f_n}{f_{n-1}} \leq \xi_n^0, \quad n = \overline{1, N}. \quad (4.74)$$

The Theorem 4.1 and inequalities (4.74) prove the Lemma 4.4.

5 Application to Mathematical Finance.

Due to Corollary 3.3, we can give the following definition of fair price of contingent claim f_N relative to a convex set of equivalent measures M .

Definition 5.1 Let f_N , $N < \infty$, be a \mathcal{F}_N -measurable integrable relative to a convex set of equivalent measures M random value such that for some $0 \leq \alpha_0 < \infty$ and $\xi_0 \in A_0$

$$P(f_N - \alpha_0 E^P\{\xi_0 | \mathcal{F}_N\} \leq 0) = 1. \quad (5.1)$$

Denote $G_{\alpha_0} = \{\alpha \in [0, \alpha_0], \exists \xi_\alpha \in A_0, P(f_N - \alpha E^P\{\xi_\alpha | \mathcal{F}_N\} \leq 0) = 1\}$. We call

$$f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha \quad (5.2)$$

a fair price of contingent claim f_N relative to a convex set of equivalent measures M , if there exists $\zeta_0 \in A_0$ and a sequences $\alpha_n \in [0, \alpha_0]$, $\xi_{\alpha_n} \in A_0$, satisfying conditions $\alpha_n \rightarrow f_0$, $\xi_{\alpha_n} \rightarrow \zeta_0$ by probability, as $n \rightarrow \infty$, and such that

$$P(f_N - \alpha_n E^P\{\xi_{\alpha_n} | \mathcal{F}_N\} \leq 0) = 1, \quad n = \overline{1, \infty}. \quad (5.3)$$

Theorem 5.1 Let the set A_0 be uniformly integrable one relative to every measure $P \in M$. Suppose that for a nonnegative \mathcal{F}_N -measurable integrable relative to every measure $P \in M$ contingent claim f_N , $N < \infty$, there exist $\alpha_0 < \infty$ and $\xi_0 \in A_0$ such that

$$P(f_N - \alpha_0 E^P\{\xi_0 | \mathcal{F}_N\} \leq 0) = 1, \quad (5.4)$$

then a fair price f_0 of contingent claim f_N exists. For f_0 the inequality

$$\sup_{P \in M} E^P f_N \leq f_0 \quad (5.5)$$

is valid. If a supermartingale $\{f_m = \text{ess sup}_{P \in M} E^P\{f_N | \mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is local regular one, then $f_0 = \sup_{P \in M} E^P f_N$.

Proof. If $f_0 = \alpha_0$, then Theorem 5.1 is proved. Suppose that $f_0 < \alpha_0$. Then there exists a sequence $\alpha_n \rightarrow f_0$, and $\xi_{\alpha_n} \in A_0$, $n \rightarrow \infty$, such that

$$P(f_N - \alpha_n E^P\{\xi_{\alpha_n} | \mathcal{F}_N\} \leq 0) = 1, \quad P \in M. \quad (5.6)$$

Due to uniform integrability A_0 we obtain

$$1 = \lim_{n \rightarrow \infty} \int_{\Omega} \xi_{\alpha_n} dP = \int_{\Omega} \zeta_0 dP, \quad P \in M. \quad (5.7)$$

Using again uniform integrability and going to the limit in (5.6) we obtain

$$P(f_N - f_0 E^P\{\zeta_0 | \mathcal{F}_N\} \leq 0) = 1, \quad P \in M. \quad (5.8)$$

From the inequality $f_N - f_0 E^P\{\zeta_0 | \mathcal{F}_N\} \leq 0$ it follows inequality (5.5). If $f_m = \text{ess sup}_{P \in M} E^P\{f_N | \mathcal{F}_m\}$, $m = \overline{0, N}$, is a local regular supermartingale, then

$$f_m = M_m - g_m, \quad m = \overline{0, N}, \quad g_0 = 0, \quad (5.9)$$

where a martingale M_m , $m = \overline{0, N}$, is a nonnegative and $E^P M_m = \sup_{P \in M} E^P f_N$.

Introduce into consideration a random value $\xi_0 = \frac{M_N}{f_0}$, $\hat{f}_0 = \sup_{P \in M} E^P f_N$. Then ξ_0 belongs to the set A_0 and

$$P(f_N - \hat{f}_0 E^P\{\xi_0 | \mathcal{F}_N\} \leq 0) = 1. \quad (5.10)$$

From this it follows that $f_0 = \sup_{P \in M} E^P f_N$.

Let us prove that f_0 is a fair price for some evolution of risk and non risk assets. Suppose that evolution of risk asset is given by the law $S_m = f_0 M^P\{\zeta_0 | \mathcal{F}_m\}$, $m = \overline{0, N}$, and evolution of non risk asset is given by the formula $B_m = 1$, $m = \overline{0, N}$.

As proved above, for $f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha$ there exists $\zeta_0 \in A_0$ such that the inequality

$$f_N - f_0 E^P\{\zeta_0 | \mathcal{F}_N\} \leq 0$$

is valid. Let us put

$$\begin{aligned} f_m^0 &= f_0 E^P\{\zeta_0 | \mathcal{F}_m\}, \quad P \in M, \\ \bar{f}_m &= \begin{cases} 0, & m < N, \\ f_N - f_0 E^P\{\zeta_0 | \mathcal{F}_m\}, & m = N. \end{cases} \end{aligned}$$

It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0$, $m = \overline{0, N}$. Therefore, the supermartingale

$$f_m^0 + \bar{f}_m = \begin{cases} f_0 E^P\{\zeta_0 | \mathcal{F}_m\}, & m < N, \\ f_N, & m = N, \end{cases}$$

is a local regular one. It is evident that

$$f_m^0 + \bar{f}_m = M_m - g_m, \quad m = \overline{0, N},$$

where

$$M_m = f_0 E^P\{\zeta_0 | \mathcal{F}_m\}, \quad m = \overline{0, N},$$

$$g_m = 0, \quad m = \overline{0, N-1},$$

$$g_N = f_0 E^P \{\zeta_0 | \mathcal{F}_N\} - f_N.$$

For martingale $\{M_m\}_{m=0}^N$ the representation

$$M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, \quad m = \overline{0, N},$$

is valid, where $H_i = 1$, $i = \overline{1, N}$. Let us consider a trading strategy $\pi = \{\bar{H}_m^0, \bar{H}_m\}_{m=0}^N$, where

$$\bar{H}_0^0 = f_0, \quad \bar{H}_m^0 = M_m - H_m S_m, \quad m = \overline{1, N}, \quad \bar{H}_0 = 0, \quad \bar{H}_m = H_m, \quad m = \overline{1, N}.$$

It is evident that \bar{H}_m^0, \bar{H}_m are \mathcal{F}_{m-1} measurable and the trading strategy π satisfy self-financed condition

$$\Delta \bar{H}_m^0 + \Delta \bar{H}_m S_{m-1} = 0.$$

Moreover, a capital corresponding to the self-financed trading strategy π is given by the formula

$$X_m^\pi = \bar{H}_m^0 + \bar{H}_m S_m = M_m.$$

Herefrom, $X_0^\pi = f_0$. Further,

$$X_N^\pi = f_N + g_N \geq f_N.$$

The last proves the Theorem 5.1. From (5.8) and Corollary 3.3 the Theorem 5.2 follows.

Theorem 5.2 *Suppose that the set A_0 contains only $1 \leq k < \infty$ linear independent elements ξ_1, \dots, ξ_k . If there exist $\xi_0 \in T$ and $\alpha_0 \geq 0$ such that*

$$P(f_N - \alpha_0 E^P \{\xi_0 | \mathcal{F}_N\} \leq 0) = 1, \quad P \in M, \quad (5.11)$$

where

$$T = \{\xi \geq 0, \quad \xi = \sum_{i=1}^k \alpha_i \xi_i, \quad \alpha_i \geq 0, \quad i = \overline{1, k}, \quad \sum_{i=1}^k \alpha_i = 1\}, \quad (5.12)$$

then a fair price f_0 of contingent claim $f_N \geq 0$ exists, where f_N is \mathcal{F}_N measurable and integrable relative to every measure $P \in M$, $N < \infty$.

Proof. The proof is evident, as the set T is uniformly integrable relative to every measure from M .

Corollary 5.1 *On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_m on it, let $\{f_m, \mathcal{F}_m\}_{m=0}^N$ be a non negative local regular supermartingale relative to a convex set of equivalent measures M . If the set A_0 is uniformly integrable relative to every measure $P \in M$, then the fair price of contingent claim f_N exists.*

Proof. From optional decomposition we have $f_m = M_m - g_m$, $m = \overline{0, N}$. Therefore, $P(f_N - \alpha_0 \xi_0 \leq 0) = 1$, where $\alpha_0 = E^P M_N$, $P \in M$, $\xi_0 = \frac{M_N}{E^P M_N}$. From the last it follows that conditions of Theorem 5.2 are satisfied. The Corollary 5.1 is proved.

On a probability space $\{\Omega, \mathcal{F}, P\}$ let us consider an evolution of one risk asset given by the law $\{S_m\}_{m=0}^N$, where S_m is a random value taking values in R_+ . Suppose

that \mathcal{F}_m is a filtration on $\{\Omega, \mathcal{F}, P\}$. We assume that non risk asset evolve by the law $B_m^0 = 1$, $m = \overline{1, N}$. Denote by $M^e(S)$ the set of all martingale measures being equivalent to the measure P . We assume that the set $M^e(S)$ of such martingale measures is not empty and effective market is non complete, see, for example, [3], [14], [4], [12]. So, we have that

$$E^Q\{S_m|\mathcal{F}_{m-1}\} = S_{m-1}, \quad m = \overline{1, N}, \quad Q \in M^e(S). \quad (5.13)$$

The next Theorem justify the Definition 5.1.

Theorem 5.3 *Let a contingent claim f_N be a \mathcal{F}_N -measurable integrable random value with respect to every measure from $M^e(S)$ and conditions of the Theorem 5.2 are satisfied with $\xi_i = \frac{S_i}{S_0}$, $i = \overline{0, N}$. Then there exists self-financed trade strategy π the capital evolution $\{X_m^\pi\}_{m=0}^N$ of which is a martingale relative to every measure from $M^e(S)$ satisfying conditions $X_0^\pi = f_0$, $X_N^\pi \geq f_N$, where f_0 is a fair price of contingent claim f_N .*

Proof. Due to Theorems 5.1, 5.2, for $f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha$ there exists $\zeta_0 \in A_0$ such that the inequality

$$f_N - f_0 E^P\{\zeta_0|\mathcal{F}_N\} \leq 0 \quad (5.14)$$

is valid. Let us put

$$\begin{aligned} f_m^0 &= f_0 E^P\{\zeta_0|\mathcal{F}_m\}, \quad P \in M^e(S), \\ \bar{f}_m &= \begin{cases} 0, & m < N, \\ f_N - f_0 E^P\{\zeta_0|\mathcal{F}_m\}, & m = N. \end{cases} \end{aligned}$$

It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0$, $m = \overline{0, N}$. Therefore, the supermartingale

$$f_m^0 + \bar{f}_m = \begin{cases} f_0 E^P\{\zeta_0|\mathcal{F}_m\}, & m < N, \\ f_N, & m = N, \end{cases}$$

is a local regular one. It is evident that

$$f_m^0 + \bar{f}_m = M_m - g_m, \quad m = \overline{0, N},$$

where

$$\begin{aligned} M_m &= f_0 E^P\{\zeta_0|\mathcal{F}_m\}, \quad m = \overline{0, N}, \\ g_m &= 0, \quad m = \overline{0, N-1}, \\ g_N &= f_0 E^P\{\zeta_0|\mathcal{F}_N\} - f_N. \end{aligned}$$

Due to Theorem 6.2, for martingale $\{M_m\}_{m=0}^N$ the representation

$$M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, \quad m = \overline{0, N},$$

is valid. Let us consider a trading strategy $\pi = \{\bar{H}_m^0, \bar{H}_m\}_{m=0}^N$, where

$$\bar{H}_0^0 = f_0, \quad \bar{H}_m^0 = M_m - H_m S_m, \quad m = \overline{1, N}, \quad \bar{H}_0 = 0, \quad \bar{H}_m = H_m, \quad m = \overline{1, N}.$$

It is evident that \bar{H}_m^0, \bar{H}_m are \mathcal{F}_{m-1} measurable and the trading strategy π satisfy self-financed condition

$$\Delta \bar{H}_m^0 + \Delta \bar{H}_m S_{m-1} = 0.$$

Moreover, a capital corresponding to the self-financed trading strategy π is given by the formula

$$X_m^\pi = \bar{H}_m^0 + \bar{H}_m S_m = M_m.$$

Herefrom, $X_0^\pi = f_0$. Further,

$$X_N^\pi = f_N + g_N.$$

Therefore $X_N^\pi \geq f_N$. Theorem 5.3 is proved.

In the next theorem we assume that evolutions of risk and non risk assets generate incomplete market [3], [14], [4], [12], that is, the set of martingale measures contains more than one element.

Theorem 5.4 *Let an evolution $\{S_m\}_{m=0}^N$ of risk asset satisfy conditions $P(D_m^1 \leq S_m \leq D_m^2) = 1$, $D_{m-1}^1 \geq D_m^1 > 0$, $D_{m-1}^2 \leq D_m^2 < \infty$, $m = \overline{1, N}$, and let non risk asset evolution be deterministic one given by the law $\{B_m\}_{m=0}^N$, $B_m = 1$, $m = \overline{0, N}$. The fair price of standard European call option with payment function $f_N = (S_N - K)^+$ is given by the formula*

$$f_0 = \begin{cases} S_0(1 - \frac{K}{D_N^2}), & K \leq D_N^2, \\ 0, & K > D_N^2. \end{cases} \quad (5.15)$$

The fair price of standard European put option with payment function $f_N = (K - S_N)^+$ is given by the formula

$$f_0 = \begin{cases} K - D_N^1, & K \geq D_N^1, \\ 0, & K < D_N^1. \end{cases} \quad (5.16)$$

Proof. In the Theorem 5.4 conditions the set of equations $E^P \zeta = 1$, $\zeta \geq 0$, has solutions $\zeta_i = \frac{S_i}{S_0}$, $i = \overline{0, N}$. It is evident that $\alpha_0 = S_0$ and $\zeta_N = \frac{S_N}{S_0}$, since

$$\frac{(S_N - K)^+}{B_N} - \alpha_0 \frac{S_N}{S_0} \leq 0, \quad \omega \in \Omega.$$

Let us prove the needed formula. Consider the inequality

$$(S_N - K) - \alpha \sum_{i=0}^N \gamma_i \frac{S_i}{S_0} \leq 0, \quad \gamma \in V_0, \quad (5.17)$$

where $V_0 = \{\gamma = \{\gamma_i\}_{i=0}^N, \gamma_i \geq 0, \sum_{i=0}^N \gamma_i = 1\}$. Or,

$$S_N \left(1 - \frac{\alpha \gamma_N}{S_0}\right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \quad (5.18)$$

Suppose that α satisfies inequality

$$1 - \frac{\alpha}{S_0} > 0. \quad (5.19)$$

If α satisfies additionally the equality

$$D_N^2 \left(1 - \frac{\alpha \gamma_N}{S_0} \right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D_i^1}{S_0} = 0, \quad (5.20)$$

then for all $\omega \in \Omega$ (5.18) is valid. From (5.20) we obtain for α

$$\alpha = \frac{S_0(D_N^2 - K)}{(D_N^2 \gamma_N + \sum_{i=0}^{N-1} \gamma_i D_i^1)}. \quad (5.21)$$

If $D_N^2 - K > 0$, then

$$\inf_{\gamma \in V_0} \frac{S_0(D_N^2 - K)}{(D_N^2 \gamma_N + \sum_{i=0}^{N-1} \gamma_i D_i^1)} = \frac{S_0(D_N^2 - K)}{D_N^2}, \quad (5.22)$$

since $D_N^2 \geq D_i^1$. From here we obtain

$$f_0 = S_0 \left(1 - \frac{K}{D_N^2} \right). \quad (5.23)$$

It is evident that $\alpha = f_0$ satisfies inequality (5.19).

If $D_N^2 - K \leq 0$, then $S_N - K \leq 0$ and from (5.17) we can put $\alpha = 0$. Then, the formula (5.18) is valid for all $\omega \in \Omega$.

Let us prove the formula (5.16) for standard European put option. If $S_N \leq K$ it is evident that $\alpha_0 = K$, and $\zeta_0 = 1$, since

$$(K - S_N) - \alpha_0 \leq 0, \quad \omega \in \Omega.$$

Let us prove the needed formula. Consider the inequality

$$(K - S_N)^+ - \alpha \sum_{i=0}^N \gamma_i \frac{S_i}{S_0} \leq 0, \quad \gamma \in V_0. \quad (5.24)$$

Or, for $S_N \leq K$

$$-S_N \left(1 + \frac{\alpha \gamma_N}{S_0} \right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \quad (5.25)$$

If α is a solution of the equality

$$-D_N^1 \left(1 + \frac{\alpha \gamma_N}{S_0} \right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D_i^1}{S_0} = 0, \quad (5.26)$$

then for all $\omega \in \Omega$ (5.25) is valid. From (5.26) we obtain for α

$$\alpha = \frac{S_0(K - D_N^1)}{\sum_{i=0}^N \gamma_i D_i^1}. \quad (5.27)$$

Therefore,

$$\inf_{\gamma \in V_0} \frac{S_0(K - D_N^1)}{\sum_{i=0}^N \gamma_i D_i^1} = K - D_N^1, \quad (5.28)$$

since $D_i^1 \leq S_0$, $i = \overline{1, N}$, $D_0^1 = S_0$. From here we obtain

$$f_0 = K - D_N^1. \quad (5.29)$$

If $D_N^1 - K > 0$, then $S_N - K > 0$ and from (5.24) we can put $\alpha = 0$. Then, (5.25) is valid for all $\omega \in \Omega$. The Theorem 5.4 is proved.

6 Some auxiliary results.

On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration \mathcal{F}_n on it, let us consider a convex set of equivalent measures M . Suppose that ξ_1, \dots, ξ_d is a set of random values belonging to the set A_0 . Introduce d martingales relative to a set of measures M $\{S_n^i, \mathcal{F}_n\}_{n=0}^\infty$, $i = \overline{1, d}$, where $S_n^i = E^P\{\xi_i | \mathcal{F}_n\}$, $i = \overline{1, d}$, $P \in M$. Denote by $M^e(S)$ a set of all equivalent to a measure $P \in M$ martingale measures, that is, $Q \in M^e(S)$ if

$$E^Q\{S_n | \mathcal{F}_{n-1}\} = S_{n-1}, \quad E^Q|S_n| < \infty, \quad Q \in M^e(S), \quad n = \overline{1, \infty}.$$

It is evident that $M \subseteq M^e(S)$ and $M^e(S)$ is a convex set. Denote by P_0 a certain fixed measure from $M^e(S)$ and let $L^0(R^d)$ be a set of finite valued random values on a probability space $\{\Omega, \mathcal{F}, P_0\}$, taking values in R^d .

Let H^0 be a set of finite valued predictable processes $H = \{H_n\}_{n=1}^N$, where $H_n = \{H_n^i\}_{i=1}^d$ takes values in R^d and H_n is \mathcal{F}_{n-1} -measurable. Introduce into consideration a set of random values

$$K_N^1 = \{\xi \in L^0(R^1), \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, H \in H^0\}, \quad N < \infty, \quad (6.1)$$

$$\Delta S_k = S_k - S_{k-1}, \quad \langle H_k, \Delta S_k \rangle = \sum_{s=1}^d H_k^s (S_k^s - S_{k-1}^s).$$

Lemma 6.1 *The set of random values K_N^1 is a closed subset in the set of finite valued random values $L^0(R^1)$ relative to convergence by measure $P \in M$.*

The proof of the Lemma 6.1 see, for example, [3].

Introduce into consideration a subset

$$V^0 = \{H \in H^0, \|H_n\| < \infty, n = \overline{1, N}\}$$

of the set H^0 , where $\|H_n\| = \sup_{\omega \in \Omega} \sum_{i=1}^d |H_n^i|$. Let K_N be a subset of the set K_N^1

$$K_N = \{\xi \in L^0(R^1), \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, H \in V^0\}.$$

Denote also a set

$$C = \{k - f, \ k \in K_N, \ f \in L_+^\infty(\Omega, \mathcal{F}, P_0)\},$$

where $L_+^\infty(\Omega, \mathcal{F}, P_0)$ is a set of bounded nonnegative random values. Let \bar{C} be a closure of C in $L^1(\Omega, \mathcal{F}, P_0)$ metrics.

Lemma 6.2 *If $\zeta \in \bar{C}$ and such that $E^{P_0}\zeta = 0$, then for ζ the representation*

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle$$

is valid for a certain finite valued predictable process $H = \{H_n\}_{n=1}^N$.

Proof. If $\zeta \in K_N$, then Lemma 6.2 is proved. Suppose that $\zeta \in \bar{C}$, then there exists a sequence $k_n - f_n$, $k_n \in K_N$, $f_n \in L_+^\infty(\Omega, \mathcal{F}, P_0)$ such that $\|k_n - f_n - \zeta\|_{P_0} \rightarrow 0$, $n \rightarrow \infty$, where $\|g\|_{P_0} = E^{P_0}|g|$. Since $|E^{P_0}(k_n - f_n - \zeta)| \leq \|k_n - f_n - \zeta\|_{P_0}$, we have $E^{P_0}f_n \leq \|k_n - f_n - \zeta\|_{P_0}$. From here we obtain $\|k_n - \zeta\|_{P_0} \leq 2\|k_n - f_n - \zeta\|_{P_0}$. Therefore, $k_n \rightarrow \zeta$ by measure P_0 . On the basis of the Lemma 6.1, a set

$$K_N^1 = \{\xi \in L^0(R^1), \ \xi = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle, \ H \in H^0\}, \ \langle H_k, \Delta S_k \rangle = \sum_{i=1}^d H_k^i (S_k^i - S_{k-1}^i)$$

is a closed subset of $L^0(R^1)$ relative to convergence by measure P_0 . From this fact, we obtain the proof of Lemma 6.2, since there exists finite valued predictable process $H \in H^0$ such that for ζ the representation

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle$$

is valid.

Theorem 6.1 *Let $E^Q|\zeta| < \infty$, $Q \in M^e(S)$. If for every $Q \in M^e(S)$, $E^Q\zeta = 0$, then there exists finite valued predictable process H such that for ζ the representation*

$$\zeta = \sum_{k=1}^N \langle H_k, \Delta S_k \rangle \tag{6.2}$$

is valid.

Proof. If $\zeta \in \bar{C}$, then (6.2) follows from Lemma 6.2. So, let ζ does not belong to \bar{C} . As in Lemma 6.2, \bar{C} is a closure of C in $L^1(\Omega, \mathcal{F}, P_0)$ metrics for the fixed measure P_0 . The set \bar{C} is a closed convex set in $L^1(\Omega, \mathcal{F}, P_0)$. Consider the other convex closed set that consists from one element ζ . Due to Han – Banach Theorem, there exists a linear continuous functional l_1 , which belongs to $L^\infty(\Omega, \mathcal{F}, P_0)$, and real numbers $\alpha > \beta$ such that

$$l_1(\xi) = \int_{\Omega} \xi(\omega) q(\omega) dP_0, \quad q(\omega) \in L^\infty(\Omega, \mathcal{F}, P_0), \tag{6.3}$$

and inequalities $l_1(\zeta) > \alpha$, $l_1(\xi) \leq \beta$, $\xi \in \bar{C}$, are valid. Since \bar{C} is a convex cone we can put $\beta = 0$. From condition $l_1(\xi) \leq 0$, $\xi \in \bar{C}$ we have $l_1(\xi) = 0$, $\xi \in K_N^1 \cap L^1(\Omega, \mathcal{F}, P_0)$. From (6.3) and inclusions $\bar{C} \supset C \supset -L^\infty(\Omega, \mathcal{F}, P_0)$ we have $q(\omega) \geq 0$. Introduce a measure

$$Q^*(A) = \int_A q(\omega) dP_0 \left[\int_\Omega q(\omega) dP_0 \right]^{-1}.$$

Then, we have

$$\int_\Omega \xi(\omega) dQ^* = 0, \quad \xi \in K_N^1 \cap L^1(\Omega, \mathcal{F}, P_0). \quad (6.4)$$

Let us choose $\xi = \chi_A(\omega)(S_i^j - S_{i-1}^j)$, $A \in \mathcal{F}_{i-1}$, where $\chi_A(\omega)$ is an indicator of a set A . We obtain

$$\int_A (S_i^j - S_{i-1}^j) dQ^* = 0, \quad A \in \mathcal{F}_{i-1}.$$

So, Q^* is a martingale measure that belongs to the set $M^a(S)$, which is a set of absolutely continuous martingale measures. Let us choose $Q \in M^e(S)$ and consider a measure $Q_1 = (1 - \gamma)Q + \gamma Q^*$, $0 < \gamma < 1$. A measure $Q_1 \in M^e(S)$ and, moreover, $E^{Q_1}\zeta = \gamma E^{Q^*}\zeta > 0$. We come to the contradiction with conditions of Theorem 6.1, since for $Q \in M^e(S)$, $E^Q\zeta = 0$. So, $\zeta \in \bar{C}$, and in accordance with the Lemma 6.2, for ζ the declared representation in Theorem 6.1 is valid.

Theorem 6.2 *For every martingale $\{M_n, \mathcal{F}_n\}_{n=0}^\infty$ relative to the set of measures $M^e(S)$, there exists a predictable random process H such that for M_n , $n = \overline{0, \infty}$, the representation*

$$M_n = M_0 + \sum_{i=1}^n \langle H_i, \Delta S_i \rangle, \quad n = \overline{1, \infty}, \quad (6.5)$$

is valid.

Proof. For fixed natural $N \geq 1$, let us consider random value $M_N - M_0 = \zeta$. Since

$$E^Q|\zeta| < \infty, \quad E^Q\zeta = 0, \quad Q \in M^e(S),$$

then ζ satisfies conditions of Theorem 6.1 and, therefore, belongs to \bar{C} , so, there exists a sequence $k_n = \sum_{i=1}^N \langle H_i^n, \Delta S_i \rangle \in K_N$ such that

$$\int_\Omega |k_n - \zeta| dP_0 \rightarrow 0, \quad n \rightarrow \infty.$$

From here, we obtain

$$\int_\Omega |E^{P_0}\{(k_n - \zeta)|\mathcal{F}_m\}| dP_0 \leq \int_\Omega |k_n - \zeta| dP_0 \rightarrow 0, \quad n \rightarrow \infty.$$

But $E^{P_0}\{k_n|\mathcal{F}_m\} = \sum_{i=1}^m \langle H_i^n, \Delta S_i \rangle$. Hence, we obtain that as $\sum_{i=1}^m \langle H_i^n, \Delta S_i \rangle$ and $\sum_{i=1}^N \langle H_i^n, \Delta S_i \rangle$ converges by measure P_0 to $E^{P_0}\{\zeta|\mathcal{F}_m\}$ and ζ , correspondingly. There exists a subsequence n_k such that H^{n_k} converges everywhere to predictable process H . From here, we have $\zeta = \sum_{i=1}^N \langle H_i, \Delta S_i \rangle$ and $E^{P_0}\{\zeta|\mathcal{F}_m\} = \sum_{i=1}^m \langle H_i, \Delta S_i \rangle$. It proves that for all $m < N$

$$M_m = M_0 + \sum_{i=1}^m \langle H_i, \Delta S_i \rangle.$$

Theorem 6.2 is proved.

7 Conclusions.

In the paper, we generalize Doob decomposition for supermartingales relative to one measure onto the case of supermartingales relative to a convex set of equivalent measures. For supermartingales relative to one measure for continuous time Doob's result was generalized in papers [18] [19].

Section 2 contains the auxiliary statements giving sufficient conditions of the existence of maximal element in a maximal chain, of the existence of nonzero non-decreasing process such that the sum of a supermartingale and this process is again a supermartingale relative to a convex set of equivalent measures needed for the main Theorems. In Theorem 2.2 we give sufficient conditions of the existence of the optional Doob decomposition for the special case as the set of measures is generated by finite set of equivalent measures with bounded as below and above the Radon - Nicodym derivatives. After that, we introduce the notion of a regular supermartingale. Theorem 2.3 describes regular supermartingales. In Theorem 2.4 we give the necessary and sufficient conditions of regularity of supermartingales. After that we introduce a notion of local regular supermartingale. At last, we prove Theorem 2.6 asserting that if the optional decomposition for a supermartingale is valid, then it is local regular one. Essentially, Theorem 2.6 and 2.7 give the necessary and sufficient conditions of local regularity of supermartingale.

In section 3 we prove auxiliary statements needed for the description of local regular supermartingales. The notion of a local regular supermartingale relative to a convex set of equivalent measures is equivalent to the existence of non negative adapted process such that the equalities (2.71) are valid. Since the existence of optional decomposition for supermartingale and existence of adapted non negative process entering (2.71) are equivalent ones, then it would seem to obtain new information from the set of equation (2.71) is impossible. As it was found, this new formulation are proved to be fruitful, since it turned out to describe the structure of all local regular supermartingales relative to a convex set of equivalent measures. For this purpose we investigate the structure of supermartingales of special types relative to a convex set of equivalent measures, generated by a certain finite set of equivalent measures. The main result of this investigation is the Lemma 3.7, which allowed us to prove Lemma 3.8, stating sufficient conditions of existence of a martingale on a measurable space with respect to a convex set of equivalent measures generated by finite set of equivalent measures. The existence of non trivial random value satisfying conditions (3.20) is sufficient condition for the existence of non trivial martingale with respect to a convex set of equivalent measures, generated by finite set of equivalent measures. Theorem 3.1 describes all local regular non negative supermartingales of special type (3.21) relative to constructed above set of equivalent measures.

In the Theorem 3.2 we give sufficient conditions of the existence of local regular martingale relative to arbitrary set of equivalent measures and arbitrary filtration. If time interval is finite these conditions are also necessary. After that, we present in Theorem 3.3 important construction of local regular supermartingales which we sum up in Corollary 3.2. Theorem 3.4 proves that every non negative uniformly integrable supermartingale belongs to described class (3.38) of local regular supermartingales.

Section 4 contains the Theorem 4.1 giving a variant of the necessary and sufficient conditions of local regularity of non negative supermartingale relative to a convex set of equivalent measures. In subsection 1 the Definition 4.1 determine a class of complete set of equivalent measures. The Lemma 4.1 guarantee a bound (4.12) for all non negative random values allowing us to prove the Theorem 4.2, stating that for every non negative supermartingale optional decomposition is valid. In subsection 2 we extend the results of subsection 1 onto the case as a space of elementary events is countable. At last, subsection 3 contains the generalization of the result obtained in subsection 2 onto the case of arbitrary space of elementary events. We prove that for every non negative supermartingale optional decomposition is valid.

Corollary 3.3 of the Section 5 contains important construction of the local regular supermartingales playing important role in definition of fair price of contingent claim relative to a convex set of equivalent measures. The Definition 5.1 is fundamental for evaluation of risk in incomplete markets. Theorem 5.1 gives sufficient conditions of the existence of fair price of contingent claim relative to a convex set of equivalent measures. It also gives sufficient conditions when defined fair price coincides with classical value. In the Theorem 5.2 simple conditions of the existence of fair price of contingent claim are given. In Theorem 5.3 we prove the existence of self-financed trading strategy confirming a Definition 5.1 of fair price as parity between long and short positions in contracts. As application of the result obtained we prove Theorem 5.4, where the formulas for standard European call and put options in incomplete market we present. Section 6 contains auxiliary results needed for previous sections.

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